Appendix.

Definition of $\mathfrak{W}$ and $\mathfrak{D}$.

Let v be a place (of $\mathbb{Q}$) and let K be a finite Galois extension of $\mathbb{Q}_v$, then we have an exact sequence

$$K^\times \longrightarrow W_{K/\mathbb{Q}_v} \longrightarrow \mathrm{Gal}(K/\mathbb{Q}_v),$$

defined by a splitting $d_\sigma \in W_{K/\mathbb{Q}_v}$ ($\sigma \in \mathrm{Gal}(K/\mathbb{Q}_v)$) where $d_\rho d_\sigma d_{\rho\sigma}^{-1} = d_{\rho,\sigma}$ - a 2-cocycle in the fundamental class of $K/\mathbb{Q}_v$, of course $d_\sigma k\, d_\sigma^{-1} = \sigma(k)$ for $k \in K^\times$. The sequence is determined up to an isomorphism which in turn is determined up to conjugation by an element of $K^\times$.

If we choose an algebraic closure $\bar{\mathbb{Q}}_v$ of $\mathbb{Q}_v$ containing K, we have, by forward and backward transform, a gerb

$$\mathbb{G}_m(\bar{\mathbb{Q}}_v) \longrightarrow \mathfrak{D}^K \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v).$$

For $K \subset K'$ ($\subset \bar{\mathbb{Q}}_v$) we have a natural homomorphism $\mathfrak{D}^{K'} \longrightarrow \mathfrak{D}^K$ (determined up to conjugation by an element of $\mathbb{G}_m(\bar{\mathbb{Q}}_v)$) given by $x \mapsto x^{[K':K]}$ (on the kernel) and $d'_\sigma \mapsto c_\sigma d_\sigma$ if $(d'_{\rho,\sigma})^{[K':K]}/d_{\rho,\sigma} = c_\rho\, \rho(c_\sigma) c_{\rho\sigma}^{-1}$, and therefore we have a limit $\mathfrak{D}^v = \varprojlim_K \mathfrak{D}^K$.
Of course $\mathfrak{D}^\infty = \mathfrak{W}$: $\mathbb{G}_m(\mathbb{C}) \longrightarrow W_{\mathbb{R}} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Definition of $\mathcal{Q}$ and $\zeta_\infty : \mathcal{W} \to \mathcal{Q}$, $\zeta_p : \mathcal{D} \to \mathcal{Q}$ and $\zeta_\ell : \mathcal{G}_\ell \to \mathcal{Q}$.

Let p be a prime number. We choose an algebraic closure $\mathbb{C}$ of $\mathbb{R}$ and $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we choose imbeddings $\bar{\mathbb{Q}} \to \mathbb{C}$ and $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Let L ($\subset \bar{\mathbb{Q}}$) be a finite Galois extension of $\mathbb{Q}$ and let $\tilde{v}$ be the place of L over ∞ defined by $L \subset \bar{\mathbb{Q}} \to \mathbb{C}$ and $\mathfrak{p}$ be the place of L over p defined by $L \subset \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$.

Let $m \in \mathbb{N}$ and $q = p^m$. The set

$$
\begin{array}{rl}
Y(L,m) = \Big\{ \pi \in L^\times \,\Big|\, & \text{for each place } \bar{v} \text{ of } L \text{ over } \infty \text{ is } \Big|\prod\limits_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} \sigma\pi \Big|^{[L_{\bar{v}}:\mathbb{R}]} = q^{a[L:\mathbb{Q}]} \\
 & \qquad \text{for some } a \in \mathbb{Z} \\
 & \text{———— '' ———— } p \text{ is } \Big|\prod\limits_{\sigma \in \mathrm{Gal}(L_{\bar{v}}/\mathbb{Q}_p)} \sigma\pi \Big| = q^b \\
 & \qquad \text{for some } b \in \mathbb{Z} \\
 & \text{———— '' ————} \ \ell \neq p \text{ is } \pi \text{ an unit} \Big\}
\end{array}
$$

is a subgroup of $L^\times$ and $Y^*(L,m) = Y(L,m)/\{\text{units in } Y(L,m)\}$ is a finitely generated free group on which $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts. Let $Q(L,m)$ be the corresponding $\mathbb{Q}$-torus (that is $X^*(Q(L,m)) = Y^*(L,m)$), and let ν_∞, $\nu_p \in X_*(Q(L,m))$ be defined by

$\langle \nu_\infty, \chi_\pi \rangle$ = the a in the condition for $\bar{v} = \tilde{v}$

$\langle \nu_p, \chi_\pi \rangle$ = the b in the condition for $\bar{v} = \mathfrak{p}$,

for any $\pi \in Y(L,m)$ – here χ_π is the character of $Q(L,m)$ associated to π.

We choose imbeddings of exact sequences

$$
\begin{tikzcd}
L_{\tilde{v}}^\times \arrow[r] \arrow[d] & W_{L_{\tilde{v}}/\mathbb{R}} \arrow[r] \arrow[d] & \mathrm{Gal}(L_{\tilde{v}}/\mathbb{R}) \arrow[d] & (\infty) \\
C_L \arrow[r] & W_{L/\mathbb{Q}} \arrow[r] & \mathrm{Gal}(L/\mathbb{Q}) & (\mathbb{Q})
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
L_{\mathfrak{p}}^\times \arrow[r] \arrow[d] & W_{L_{\mathfrak{p}}/\mathbb{Q}_p} \arrow[r] \arrow[d] & \mathrm{Gal}(L_{\mathfrak{p}}/\mathbb{Q}_p) \arrow[d] & (p) \\
C_L \arrow[r] & W_{L/\mathbb{Q}} \arrow[r] & \mathrm{Gal}(L/\mathbb{Q}). & (\mathbb{Q})
\end{tikzcd}
$$

And for $v = \infty, p$ and $\bar{u} = \tilde{v}, \mathfrak{p}$ we choose a set S^{v} of representatives in the cosets $\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{\bar{u}}/\mathbb{Q}_{v})$ (such that $1 \in S^{v}$) and a section $\{\omega^{v}_{\tau} \mid \tau \in S^{v}\}$ of $W_{L/\mathbb{Q}} \longrightarrow \mathrm{Gal}(L/\mathbb{Q})$ on S^{v} (such that $\omega^{v}_{1} = 1$), and we define a splitting $\delta \sim \omega^{v}_{\delta}$ of $(\mathbb{Q})$ by $\omega^{v}_{\delta} = \omega^{v}_{\tau} d^{v}_{\sigma}$ if $\delta = \tau\sigma$ $(\tau \in S^{v}, \sigma \in \mathrm{Gal}(L_{\bar{u}}/\mathbb{Q}_{v}))$. If $\{A^{v}_{\rho,\sigma}\}$ is the 2-cocycle defined by this splitting, $\{A^{\infty}_{\rho,\sigma}\}$ and $\{A^{p}_{\rho,\sigma}\}$ are cohomologues, so $A^{\infty}_{\rho,\sigma}(A^{p}_{\rho,\sigma})^{-1}$ $= B_{\rho}\,\rho(B_{\sigma})B_{\rho\sigma}^{-1}$ for a 1-cochain $\{B_{\sigma}\}$ in C_L. $\nu_{v} \in X_{*}(Q(L,m))$ is left fixed by $\mathrm{Gal}(L_{\bar{u}}/\mathbb{Q}_{v})$, and we have

$$\sum_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{\bar{\infty}}/\mathbb{R})} \sigma\nu_{\infty} = - \sum_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{\mathfrak{p}}/\mathbb{Q}_{p})} \sigma\nu_{p},$$

if we let η denote this cocharacter of $Q(L,m)$, the 1-cochain $\{E_{\sigma}\}$ in $C_L \otimes X_{*}(Q(L,m))$ defined by

$$E_{\sigma} = \Big(\prod_{\tau \in S^{\infty}} (A^{\infty}_{\sigma,\tau})^{\sigma\tau\nu_{\infty}}\Big)\Big(\prod_{\tau \in S^{p}} (A^{p}_{\sigma,\tau})^{\sigma\tau\nu_{p}}\Big)B_{\sigma}^{\eta}$$

satisfies

$$E_{\rho}\,\rho(E_{\sigma})E_{\rho\sigma}^{-1} = D^{\infty}_{\rho,\sigma}\,D^{p}_{\rho,\sigma},$$

where $D^{v}_{\rho,\sigma} \in \prod_{\bar{v}|v} Q(L,m)(L_{\bar{v}})$ is defined by $D^{v}_{\rho,\sigma} = \prod_{\bar{v}|v} \tau''((d^{v}_{\rho_{\tau''},\sigma_{\tau'}})^{\nu_{v}})$, here $\tau, \tau', \tau'' \in S^{v}$ and $\rho_{\tau''}, \sigma_{\tau'} \in \mathrm{Gal}(L_{\bar{u}}/\mathbb{Q}_{v})$ are given by: τ is the element in S^{v} associated to $\bar{v}$ (that is $|\tau x|_{\bar{v}} = |x|_{\bar{u}}$), $\sigma\tau =$ $\tau'\sigma_{\tau'}$ and $\rho\tau' = \tau''\rho_{\tau''}$, τ'' also denote the isomorphism $Q(L,m)(L_{\bar{u}}) \longleftrightarrow Q(L,m)(L_{\bar{v}''})$ defined by τ''.

Now if $e_{\sigma} \in Q(L,m)(\mathbb{A}_L)$ is a lifting of E_{σ} (with respect to the projection $Q(L,m)(\mathbb{A}_L) \longrightarrow C_L \otimes X_{*}(Q(L,m))$), we have

$$e_{\rho}\,\rho(e_{\sigma})e_{\rho\sigma}^{-1}\,t_{\rho,\sigma} = D^{\infty}_{\rho,\sigma}\,D^{p}_{\rho,\sigma},$$

for a 2-cocycle $\{t_{\rho,\sigma}\}$ in $Q(L,m)(L)$, this 2-cocycle defines an exact sequence

$$Q(L,m)(L) \longrightarrow \mathfrak{Q}^{L}_{L,m} \longrightarrow \mathrm{Gal}(L/\mathbb{Q})$$

with a splitting $\sigma \mapsto t_\sigma \in \mathfrak{Q}^{L}_{L,m}$ (that is $t_\rho t_\sigma t_{\rho\sigma}^{-1} = t_{\rho,\sigma}$), and $\{e_v\}$ defines a homomorphism ζ_v of exact sequences .

$$\begin{array}{ccccc} L^{\times}_{\bar{v}} & \longrightarrow & W_{L_{\bar{v}}/\mathbb{Q}_v} & \longrightarrow & \mathrm{Gal}(L_{\bar{v}}/\mathbb{Q}_v) \\ \downarrow & & \downarrow \zeta_v & & \downarrow \\ Q(L,m)(L_{\bar{v}}) & \longrightarrow & (\mathfrak{Q}^{L}_{L,m})_{\bar{v}} & \longrightarrow & \mathrm{Gal}(L_{\bar{v}}/\mathbb{Q}_v) \end{array}$$

by ν_v on the kernel and $d_\sigma \mapsto (e_\sigma \mid Q(L,m)(L_{\bar{v}}))t_\sigma$, and, for $\ell \neq$ p and an imbedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\ell$, a splitting ζ_ℓ of

$$Q(L,m)(L_{\mathfrak{p}}) \longrightarrow (\mathfrak{Q}^{L}_{L,m})_{\mathfrak{p}} \longrightarrow \mathrm{Gal}(L_{\mathfrak{p}}/\mathbb{Q}_\ell)$$

by $\sigma \mapsto (e_\sigma \mid Q(L,m)(L_{\mathfrak{p}}))t_\sigma$, here $\mathfrak{p}$ is the prime ideal of L defined by $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\ell$.

$\mathfrak{Q}^{L}_{L,m}$ is uniquely determined up to an isomorphism which transform these local homomorphisms into equvivalent.

By forward and backward transform we have a gerb

$$Q(L,m)(\bar{\mathbb{Q}}) \longrightarrow \mathfrak{Q}^{L}_{m} \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

and local homomorphisms $\zeta_v : \mathfrak{D}^{L_{\bar{v}}} \to \mathfrak{Q}^{L}_{m}$ ($v = \infty$, p) and ζ_ℓ: $\mathfrak{G}_\ell \to \mathfrak{Q}^{L}_{m}$ (for $\ell \neq$ p).

For $L \subset L'$ ($\subset \bar{\mathbb{Q}}$) and $m \mid m'$ we have a homomorphism $\mathfrak{Q}^{L'}_{m'} \to \mathfrak{Q}^{L}_{m}$ transforming ν'_v to $[L'_{\bar{v}'} : L_{\bar{v}}]\nu_v$ ($v = \infty$, p), therefore we have a limit $\mathfrak{Q} = \varprojlim_{L,m} \mathfrak{Q}^{L}_{m}$, and local homomorphisms $\zeta_\infty : \mathfrak{W} \to \mathfrak{Q}$, $\zeta_p : \mathfrak{D} \to \mathfrak{Q}$ and $\zeta_\ell : \mathfrak{G}_\ell \to \mathfrak{Q}$ (for $\ell \neq$ p).

<u>Definition of $\xi_{\mu}^{\infty}: \mathcal{W} \longrightarrow \mathcal{G}_T$ and $\xi_{\mu}^{p}: \mathfrak{D} \longrightarrow \mathcal{G}_T$:</u>

Let v be a place (of $\mathbb{Q}$) and let $\bar{\mathbb{Q}}_v$ be an algebraic closure of $\mathbb{Q}_v$. Let T be a $\mathbb{Q}_v$-torus which splits over the Galois extension L ($\subset \bar{\mathbb{Q}}_v$) of $\mathbb{Q}_v$ and let $\mu \in X_*(T)$.

We define a homomorphism ξ_μ of exact sequences

$$\begin{array}{ccccc} L^{\times} & \longrightarrow & W_{L/\mathbb{Q}_v} & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}_v) \\ \downarrow & & \downarrow \xi_\mu & & \downarrow \\ T(L) & \longrightarrow & T(L) \rtimes \mathrm{Gal}(L/\mathbb{Q}_v) & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}_v) \end{array}$$

by $\sum_{\sigma \in \mathrm{Gal}(L/\mathbb{Q}_v)} \sigma\mu$ on the kernel and $d_\sigma \mapsto \prod_{\rho \in \mathrm{Gal}(L/\mathbb{Q}_v)} (d^{v}_{\sigma,\rho})^{\sigma\rho\mu} \rtimes \sigma$.
By forward and backward transform we have a homomorphism of gerbs $\xi_\mu : \mathfrak{D}^L \longrightarrow \mathcal{G}_T$, and by going to limit we have a homomorphism of gerbs $\xi_\mu : \mathfrak{D}^{v} \longrightarrow \mathcal{G}_T$.

Definition of $\varphi_\mu: \mathcal{Q} \longrightarrow \mathcal{G}_T$:

Let T be a $\mathbb{Q}$-torus which splits over the Galois extension L ($\subset \bar{\mathbb{Q}}$) of $\mathbb{Q}$ and let $\mu \in X_*(T)$. For $m \in \mathbb{N}$ sufficiently large we define a homomorphism $\varphi_\mu : Q(L,m) \longrightarrow T$ defined over $\mathbb{Q}$ in the following way: choose $a \in L^*$ such that $(a) = \mathfrak{p}^r$ (some $r \in \mathbb{N}$) ($\mathfrak{p}$ is the prime ideal of L defined by $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$) and $|Nm_{L_\mathfrak{p}/\mathbb{Q}_p} a| = q$ $(= p^m)$, then

$$\gamma = \prod_{\sigma \in Gal(L/\mathbb{Q})} \sigma(a)^{\sigma\mu} \quad (\in T(L))$$

belongs to $T(\mathbb{Q})$ and for $\lambda \in X^*(T)$ $\lambda(\gamma)$ belongs to $Y(L,m)$, therefore γ defines a homomorphism $X^*(T) \longrightarrow X^*(Q(L,m))$ which commute with the action of $Gal(L/\mathbb{Q})$. φ_μ is the homomorphism defined by this homomorphism of character groups.

For k ($\in \mathbb{N}$) sufficiently large we can find a section s of the projection χ: $Y(L,m) \longrightarrow X^*(Q(L,m))$ on $kX^*(Q(L,m))$ commuting with the action of $Gal(L/\mathbb{Q})$, and for n ($\in mk\mathbb{N}$) sufficiently large we can find a $\delta_n \in Q(L,m)(\mathbb{Q})$ satisfying $\chi_\pi(\delta_n) = s(k\chi_\pi)^{n/mk}$ for every $\pi \in Y(L,m)$. δ_n is not uniquely determined, but $\chi_\pi(\delta_n)\pi^{-n/m}$ is an unit for each π. $\{\delta_n^j \mid j \in \mathbb{Z}\}$ is Zariski dense in $Q(L,m)(\mathbb{Q})$.

φ_μ is characterized by $\varphi_\mu(\delta_{mn}) \equiv \gamma^n$ modulo an unit.

Now we will extend φ_μ to a homomorphism of gerbs $\varphi_\mu: \mathcal{Q} \to \mathcal{G}_T$. If

$$E_\sigma^v = \prod_{\tau \in S^v} \prod_{\rho \in Gal(L_{\mathfrak{w}}/\mathbb{Q}_v)} (A^v_{\sigma\tau,\rho})^{\sigma\tau\rho\mu} \in C_L \otimes X_*(T)$$

$(v = \infty, p)$ and

$$F = \prod_{\rho \in Gal(L/\mathbb{Q})} B_\rho^{-\rho\mu} \quad \in C_L \otimes X_*(T),$$

then E_σ^{v} belongs to $\prod_{\tilde v | v} T(L_{\tilde v})$ and we have

$$\psi_\mu(E_\sigma) = e'_\sigma \, F\sigma(F)^{-1}$$

where $e'_\sigma = E_\sigma^\infty E_\sigma^{p^{-1}}$, and if $f \in T(\mathbb{A}_L)$ is a lifting of F, then

$$\psi_\mu(e_\sigma) = s_\sigma^{-1} e'_\sigma \, f\sigma(f)^{-1},$$

where $s_\sigma \in T(L)$. The 1-cochain $\{s_\sigma\}$ satisfies $s_\rho \rho(s_\sigma) s_{\rho\sigma}^{-1} = \psi_\mu(t_{\rho,\sigma})$ and we define the remaining part of ψ_μ (on $\mathfrak{D}^L_{L,m}$) by $t_\sigma \mapsto s_\sigma \times \sigma$, by going to limits we have a homomorphism $\psi_\mu : \mathfrak{D} \longrightarrow \mathcal{G}_T$ of gerbs, it is determined up to composition with an automorphism of $\mathcal{G}_T$ which is locally equvivalent to the identical automorphism.

We have equvivalences

$$\psi_\mu \circ \zeta_\infty \sim \xi^\infty_\mu \ , \quad \psi_\mu \circ \zeta_p \sim \xi^p_{-\mu} \text{ and } \quad \psi_\mu \circ \zeta_\ell \sim \xi_\ell \ (\ell \neq p),$$

because

$$e'_\sigma \mid T(L_{\tilde v}) = \prod_{\rho \in \mathrm{Gal}(L_{\tilde v}/\mathbb{R})} (A^\infty_{\sigma,\rho})^{\sigma\rho\mu}$$

$$e'_\sigma \mid T(L_{\tilde p}) = \prod_{\rho \in \mathrm{Gal}(L_{\tilde p}/\mathbb{Q}_p)} (A^p_{\sigma,\rho})^{-\sigma\rho\mu}$$

and $$e'_\sigma \mid T(L_{\tilde\ell}) = 1 \text{ for } \tilde\ell | \ell, \ \ell \neq p .$$

<u>Definition of $\mathcal{P}$ and $\mathcal{Q} \to \mathcal{P}$.</u>

If we in the definition of $Y(L,m)$ figuring in the definition of $\mathcal{Q}$ replace the quantity

$$\left|\prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} \sigma\pi\right|^{[L:\mathbb{Q}]^{-1}} \quad \text{by} \quad \left|\prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{R})} \sigma\pi\right|^{[L_v:\mathbb{R}]^{-1}}$$

and in the definition of $Y^*(L,m)$ replace

$$\{\text{units in } Y(L,m)\} \quad \text{by} \quad \{\text{roots of unity in } Y(L,m)\},$$

then we get a new exact sequence and a homomorphism:

$$\begin{tikzcd}
Q(L,m)(L) \arrow[r] \arrow[d] & \mathcal{Q}^{L}_{L,m} \arrow[r] \arrow[d] & \mathrm{Gal}(L/\mathbb{Q}) \arrow[d] \\
P(L,m)(L) \arrow[r] & \mathcal{P}^{L}_{L,m} \arrow[r] & \mathrm{Gal}(L/\mathbb{Q})
\end{tikzcd},$$

and by forward and backward transform and then going to limit, we get a gerb $\mathcal{P}$ and a homomorphism $\mathcal{Q} \to \mathcal{P}$.

A homomorphism $\psi_\mu : \mathcal{Q} \to \mathcal{G}_T$ as above factorizes through $\mathcal{Q} \to \mathcal{P}$ if $\mu \in X_*(T)$ satisfies the Serre condition:

$$(\sigma - 1)(\iota + 1)\mu = (\iota + 1)(\sigma - 1)\mu = 0$$

for each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ (ι is the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$).

The elements $\delta_n \in P(L,m)(\mathbb{Q})$ (n sufficiently large multiple of m) are now uniquely determined and $\chi_\pi(\delta_n) = \pi^{n/m}$ for $\pi \in Y(L,m)$, also $\psi_\mu | P(L,m)(\mathbb{Q})$ is characterized by $\psi_\mu(\delta_n) = \gamma^{n/m}$.