Conclusion.

The fixed prime number p in this paper is assumed to be such that E is unramified at p, K_p is hyperspecial and that S(K) has good reduction at $\mathfrak{p}$ for $\mathfrak{p}|p$. If S(K) has not good reduction at $\mathfrak{p}$, the action of $W_{E_\mathfrak{p}}$ on $H^i_{ét}(S(K), \mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})$(via $W_{E_\mathfrak{p}} \subset Gal(\bar{E}_\mathfrak{p}/E_\mathfrak{p})$, see 1.1) need not be unramified (that is, trivial on $I_{E_\mathfrak{p}}$, or factorize through $W_{E_\mathfrak{p}} \to Gal(E_\mathfrak{p}^{un}/E_\mathfrak{p}) = Gal(\bar{\kappa}/\kappa)$), therefore the action of a Frobenius of $W_{E_\mathfrak{p}}$ is not necesserily well defined, but it is on $H^i_{ét}(S(K), \mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})^{I_{W_\mathfrak{p}}}$, thus the local zeta function of $(S(K), \xi)$ at $\mathfrak{p}$ could be defined by substituting this space in the cohomology formula of 1.1.

We expect that all the local zeta functions (as well as of course the remaining part of (14) for good p) can be expressed in terms of L-functions of a form not very different from that of (14).

The Hasse-Weil zeta function of $(S(K), \xi)$ is the inverse product of the local zeta functions at all the finite places of E, and this should thus has an expression in terms of L-functions. However in order to get a more appropriate form of this expression, as well as a more appropriate form of the functional equation which we expect the zeta function to satisfy, we will multiply the Hasse-Weil zeta function by local "zeta functions" also at the infinite places of E. We will define these local zeta functions such that (14) remains true at infinity. After this we will make a bid for the final form of the expression of the zeta function in terms of L-functions and for the functional equation.

We can get an idea for the definition of the local zeta functions at infinity by studying the cohomology formula for the local zeta function at a finite place where the reduction is good (see 1.1). We do namely observe that we obtain the same cohomology groups if we first reduce $V(\mathbb{Z}/\ell^n\mathbb{Z})\otimes_{K/K_0} S(K_0) \to S(K)$ modulo $\mathfrak{p}$, $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$ acts on these cohomology groups and in the formula we interpret $\Phi_{\mathfrak{p}}$ as the Frobenius in $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$. If we base change $V(\mathbb{Z}/\ell^n\mathbb{Z})\otimes_{K/K_0} S(K_0) \to S(K)$ via an imbedding $\nu : E \to \mathbb{C}$ (an infinite place), the corresponding sheaf over $S_\nu(K)(\mathbb{C})$ $(= (S(K)\otimes_\nu \mathbb{C})(\mathbb{C}))$ appears by tensoring by $\mathbb{Z}/\ell^n\mathbb{Z}$ a locally free sheaf of $\mathbb{Z}$-modules over $S_\nu(K)(\mathbb{C})$ (see the final remark in 1.1). If we instead tensorize that sheaf by $\mathbb{Q}$, we get a locally free sheaf of $\mathbb{Q}$-vector spaces $F_{\xi,\nu}(K)$ over $S_\nu(K)(\mathbb{C})$. We can define a representation ρ_i' of $W_{\overline{\nu(E)}}$ on the $\mathbb{C}$-vector space $H^i(S_\nu(K)(\mathbb{C}), F_{\xi,\nu}(K))\otimes_{\mathbb{Q}}\mathbb{C}$ (rational cohomology) for $i = 0,1,\dots,2d$ $(d = \dim S(K))$ by letting the action of $\mathbb{C}^*$ be given by the Hodge structure (that is, by the product of the action $z \mapsto z^{-p}\bar{z}^{-q}$ on a subspace of type (p,q) and the action $z \mapsto {}^\nu\xi\circ\tau h(z)$ on ${}^\nu V_{\mathbb{C}}$ in the notation below), if $\nu(E)\subset\mathbb{R}$ the complex conjugation on $S_\nu(K)$ induces an action ι^* on cohomology mapping a subspace of type (p,q) to a subspace of type (q,p) and the action of τ on a such subspace is taken to be $(-1)^p\iota^*$ (or if we like $i^{p+q}\iota^*$). By inducing ρ_i' to $W_{\mathbb{R}}$ we get a representation ρ_i of $W_{\mathbb{R}}$. This definition is motivated by the considerations below. The zeta function of $(S(K),\xi)$ at the infinite place ν should now be defined by

$$Z(s,S_\nu(K)(\mathbb{C}),F_{\xi,\nu}(K)) = \prod_{i=1}^{2d} L(s,\rho_i)^{(-1)^{i+1}}$$

(for the definition of the L-function $L(s,\rho)$ for ρ a representation of $W_{\mathbb{R}}$ see Ta).

$S_\nu(K)$ is conjectured to be the Shimura variety associated to ${}^\nu G, {}^\nu X_\infty, {}^\nu K$ defined in the following way: Langlands has constructed an extension of the connected Serre group S^o (denoted S in 3.1)

$$S^o \longrightarrow S \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

with a continouos splitting sp: $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow S(\mathbb{A}_f)$ (see L5 or MS1 - the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on S^o defined by this extension is the algebraic action - S is the Serre group, that is, the $\mathbb{Q}$-rational pro-algebraic group associated to the neutral Tannakian category $CM_{\mathbb{Q}}$ of motives over $\mathbb{Q}$ generated by the abelian varieties over $\mathbb{Q}$ of potential CM-type, the Tate object and the Artin motives (D3) - it is conjectured that for a motive in $CM_{\mathbb{Q}}$ the action of $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the cohomology is given by the action (on the representation space) of $sp(\tau) \in S(\mathbb{A}_f)$). For $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the extension defines an element $c(\tau) \in H^1(\mathbb{Q}, S^o)$(by $\sigma \mapsto a^{-1}\sigma(a)$ if $a \in S(\bar{\mathbb{Q}})$ maps to τ), the existence of the splitting implies that $c(\tau)$ is trivial at each finite place. Let $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be such that ν is the chosen imbedding $E \longrightarrow \bar{\mathbb{Q}}$ composed with τ (recall that E is Galois) and let (T,h,1) be a special point of $S(K)(\mathbb{C})$ (see 3.1). Let T^{ad} be the image of T in G^{ad} and let $\mu^{ad} \in X_*(T^{ad})$ be the projection of $\mu = \mu_h \in X_*(T)$, then (because $T_{\mathbb{R}}$ is fundamental in $G_{\mathbb{R}}$) μ^{ad} satisfies the Serre condition $(\tau - 1)(\iota + 1)\mu^{ad} = o$ for each $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$($\iota$ is the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$), therefore there is a unique $\mathbb{Q}$-rational homomorphism $\chi: S^o \longrightarrow T^{ad}$ such that $\chi \circ \mu_o = \mu^{ad}$($\mu_o$ is the canonical cocharacter of S^o). The image of $c(\tau)$ in $H^1(\mathbb{Q}, G^{ad})$ by χ defines an inner twisting ${}^\nu G$ of G which is trivial at each finite place and determined by $(\tau\mu - \mu)(-1) \in T(\mathbb{C})$ at infinity, and ${}^\nu G$ is conjectured to be independent of the choise of τ and the special point. T is also a Cartan subgroup of ${}^{\nu}G$ and if $\tau h : \mathbb{S} \to T_{\mathbb{R}}$ is the uniquely determined $\mathbb{R}$-homomorphism for which $\mu_{\tau h} = \tau\mu$ and ${}^{\nu}X_\infty$ is the ${}^{\nu}G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to {}^{\nu}G_{\mathbb{R}}$ containing τh, then ${}^{\nu}X_\infty$ is independent of the choise of τ and the special point and ${}^{\nu}G, {}^{\nu}X_\infty$ satisfies the conditions for G, X_∞ in 1.1. If we let ${}^{\nu}K$ be the image of K by the canonical isomorphism $G(\mathbb{A}_f) \xrightarrow{\sim} {}^{\nu}G(\mathbb{A}_f)$, the Shimura variety associated to ${}^{\nu}G, {}^{\nu}X_\infty, {}^{\nu}K$ should be $S_\nu(K)$. If we twist the representation space V of ξ in the same way as G we get a rational representation ${}^{\nu}\xi$ of ${}^{\nu}G$ on ${}^{\nu}V$, and the sheaf $F_{\xi,\nu}(K)$ over $S_\nu(K)(\mathbb{C})$ is ${}^{\nu}V(\mathbb{Q}) \times_{{}^{\nu}G(\mathbb{Q}), {}^{\nu}\xi} {}^{\nu}G(\mathbb{A})/{}^{\nu}K_\infty {}^{\nu}K$ (${}^{\nu}K_\infty$ is the centralizer of τh in ${}^{\nu}G(\mathbb{R})$).

By the theory of continouos cohomology we have

$$H^i(S_\nu(K)(\mathbb{C}), F_{\xi,\nu}(K)) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\pi} H^i({}^{\nu}\mathfrak{g}_\infty, {}^{\nu}\bar{K}_\infty, {}^{\nu}\xi \otimes \pi_\infty) \otimes \pi_f^{{}^{\nu}K}, \quad (*)$$

where the sum is taken over the irreducible representations π of ${}^{\nu}G(\mathbb{A})$ which occur (discretely) in $L^2({}^{\nu}G(\mathbb{Q})Z(\mathbb{R})Z_K \backslash {}^{\nu}G(\mathbb{A}))$ (of course only those for which the action of $Z(\mathbb{R})$ is given by the character ν^{-1} and the action of Z_K is trivial, and counted with multiplicity), ${}^{\nu}\mathfrak{g}_\infty$ is the Lie algebra of ${}^{\nu}G(\mathbb{R})/Z(\mathbb{R})$ and ${}^{\nu}\bar{K}_\infty = {}^{\nu}K_\infty/Z(\mathbb{R})$ (BW, VII, Theorem 5.2). The action of $W_{\overline{\nu(E)}}$ respect this decomposition (and is trivial on $\pi_f^{{}^{\nu}K}$).

If $\varphi \in \bar{\Phi}(G)_e$ contributes to (14), $m(\Pi_\infty) \neq 0$, this implies (since φ_∞ is essentially tempered) that Π_∞ is the L-packet of discrete series representations of $G(\mathbb{R})$ associated to one of the absolutly irreducible components of $\check{\xi}$, if this component is denoted $\check{\xi}_{\Pi_\infty}$ (so that the representations in Π_∞ have the same infinitesimal character as $\check{\xi}_{\Pi_\infty}$) we have $m(\Pi_\infty) = (-1)^d \times$

the multiplicity of $\check{\xi}_{\pi_\infty}$ in $\check{\xi}$ (see 3.8). A such φ belongs to $\Phi({}^{\nu}G)_e$ for each ν (because φ_∞ is elliptic) and contributes to (*) but only to the middle cohomology (that is, i = d = dim S(K) (BW, II,Theorem 5.3 and 5.4)).

Conversely, if $\varphi \in \Phi({}^{\nu}G)_e$, it contributes at most to the middle cohomology of (*), if it contributes $\Pi(\varphi_\infty)$ is one of the above L-packets (BW, III,Theorem 5.1), therefore $m(\Pi_\infty) \neq 0$ and since $\varphi \in \Phi(G)_e$, φ contributes to (14).

The total tempered elliptic contribution to the zeta function at infinity is precisely the term $\prod\limits_{\varphi\in\Phi(G)_e}\ldots$ of (14) where π_p is replaced by $\Pi_\infty^H = \Pi(\psi_\infty)$. This is an immediately consequence of the equvivalence of representations of $W_{\mathbb{R}}$:

$$r_\nu^{H,i}\circ\psi_\infty \sim |\cdot|^{d/2}\ast\mathrm{Ind}(W_{\mathbb{R}},W_{\nu(E)},\rho_d^i\Big|\bigoplus_{\pi\in\Pi^{i}_{\nu,\infty}} H^d({}^{\nu}\mathfrak{g}_\infty,{}^{\nu}\bar{K}_\infty,{}^{\nu}\xi_{\pi_\infty}\otimes\pi)),$$

r_ν is defined in the same way as $r_{\varphi,j}$ in 1.12, that is, choose $\tau\in$ Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) such that ν is the chosen imbedding composed with τ, since τ normalizes Gal($\bar{\mathbb{Q}}/E$), $1\times\tau\in{}^{L}G^{o}\rtimes$ Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) normalizes ${}^{L}G^{o}\rtimes$ Gal($\bar{\mathbb{Q}}/E$) and if we restrict ${}^{o}r\circ\mathrm{ad}(1\times\tau)$ to ${}^{L}G^{o}\rtimes$ Gal($\mathbb{C}/\mathbb{R}$) (or if E is not real, to ${}^{L}G^{o}$ and then induce to ${}^{L}G^{o}\rtimes$ Gal($\mathbb{C}/\mathbb{R}$)) and lift to ${}^{L}G^{o}\rtimes W_{\mathbb{R}}$ we get r_ν, $|\cdot|$ is the character $z\mapsto z\bar{z}$ of $W_{\mathbb{C}}$ or $W_{\mathbb{R}}$. We shall use that the multiplicity of $\pi\in\Pi(\varphi)$ ($\varphi\in\Phi({}^{\nu}G)_e$) in $L^2({}^{\nu}G(\mathbb{Q})Z(\mathbb{R})Z_K\backslash{}^{\nu}G(\mathbb{A}))$ is $d_\varphi|\mathcal{S}_\varphi|^{-1}\sum\limits_{s\in\mathcal{S}_\varphi}\langle s,\pi\rangle$, that $r^{H,i}\big|{}^{L}H^{o}\rtimes W_{\mathbb{R}} = \bigoplus\limits_{\nu} r_\nu^{H,i}$ and that $\langle s,\Pi_\infty^{i}\rangle = \langle s,\Pi^{i}_{\nu,\infty}\rangle$.

[Proof of the above equvivalence of representations of $W_{\mathbb{R}}$: We use the terminology of 1.12 and we assume first that ν is the chosen imbedding. Let δ_o be the half sum of the positive roots of T_o in G for the order making $\Lambda_o\in X_*({}^{L}T^{o})\otimes\mathbb{R}$ dominant, and let $\gamma_o\in X^*(T_o)$ be the highest weight of $\check{\xi}_{\pi_\infty}$ w.r.t. this order. Then $\Lambda_o = \gamma_o + \delta_o$.

Let $G(\mathbb{R})^{\bullet} = T_o(\mathbb{R})\,G_{der}(\mathbb{R})^o = Z(\mathbb{R})\,G(\mathbb{R})^o$. The representation $\pi\in\Pi_\infty$ attached to $\lambda\in\mathcal{R}\lambda$ is obtained by inducing to $G(\mathbb{R})$ the discrete series representation of $G(\mathbb{R})^{\bullet}$ attached to λ. The restriction of π to $G(\mathbb{R})^{\bullet}$ is the direct sum of the representations of $G(\mathbb{R})^{\bullet}$ attached to $\mathcal{R}(G(\mathbb{R}),T_o(\mathbb{R}))\cdot\lambda$. For $\pi\neq\pi'$ these two sets of representations of $G(\mathbb{R})^{\bullet}$ are disjoint. The set of representations of $G(\mathbb{R})^{\bullet}$ attached to the set of characters $\mathcal{R}\lambda$ has the same cardinality as $\mathcal{R}\mu$. A one-to-one correspondance is established by letting $\mu = \omega\mu_{h_o}$ correspond to the representation attached to $\omega^{-1}\lambda_o$ $(= \lambda_o\circ\omega)$.

If $\varphi_\infty(\tau) = n\times\tau$ $(n\in \mathrm{Norm}_{{}^LG^o}({}^LT^o))$ we let $\bar\omega = n^{{}^LT^o}\in \mathcal{R}({}^LG^o,{}^LT^o)$, and for $\mu\in\mathcal{R}\mu$ we let $\bar\mu = \bar\omega\mu$, then $\bar\mu = \iota'\mu$ and $\bar\mu\neq\mu$ if E is real. The operator ${}^or(n)$ - denoted by $u\mapsto nu$ - transforms the weight space corresponding to μ to that corresponding to $\bar\mu$. If E is real and $\pi^{\bullet}$ is the representation of $G(\mathbb{R})^{\bullet}$ attached to $\lambda\in\mathcal{R}\lambda$, we let $\bar\pi^{\bullet}$ be that attached to $\bar\omega\lambda$ (we note that $\bar\omega\in\mathcal{R}(G(\mathbb{R}),T_o(\mathbb{R}))$(MS2, Corollary 4.3), therefore $\pi^{\bullet}$ and $\bar\pi^{\bullet}$ induce to the same representation of $G(\mathbb{R})$), if $\pi^{\bullet}$ corresponds to μ then $\bar\pi^{\bullet}$ corresponds to $\bar\mu$.

For $\mu\in\mathcal{R}\mu$ let $\mathbb{C}_\mu$ be the restriction of ${}^or\circ\varphi_\infty|_{W_{\mathbb{C}}}$ to the weight space of ${}^or$ corresponding to μ, and let $\mathbb{C}_\mu\oplus\mathbb{C}_{\bar\mu}$ be the representation of $W_{\mathbb{R}}$ given on $W_{\mathbb{C}}$ as $\mathbb{C}_\mu\oplus\mathbb{C}_{\bar\mu}$ and letting τ act as $u\oplus\bar u \mapsto \iota(n)\bar u\oplus nu$. Then we have

$$ {}^or\circ\varphi_\infty|_{W_{\mathbb{C}}} \sim \bigoplus_{\mu\in\mathcal{R}\mu}\mathbb{C}_\mu $$

and

$$r\circ\varphi_\infty \sim \begin{cases} \bigoplus_{\mu\in\mathcal{R}_\mu/\sim} (\mathbb{C}_\mu \oplus \mathbb{C}_{\bar\mu}) \text{ if } E \text{ is real} \quad (\mu'\sim\mu \Leftrightarrow \mu' = \bar\mu) \\ \bigoplus_{\mu\in\mathcal{R}_\mu} (\mathbb{C}_\mu \oplus \mathbb{C}_{\bar\mu}) \text{ if } E \text{ is not real.} \end{cases}$$

If we induce $\mathbb{C}_\mu$ to $W_{\mathbb{R}}$, we get a representation on $\mathbb{C}_\mu \oplus \mathbb{C}_\mu$: $z\in\mathbb{C}^*$ acts as $z\oplus\bar z$ and τ acts as $u\oplus u' \mapsto (-1)^d u'\oplus u$. This representation is equvivalent to $\mathbb{C}_\mu\oplus\mathbb{C}_{\bar\mu}$ (an equvivalence is given by $u\oplus u'\mapsto u\oplus nu'$). Therefore we have if E is not real:

$$r\circ\varphi_\infty \sim \operatorname{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, {}^{o}r\circ\varphi_\infty|_{W_{\mathbb{C}}}).$$

If π° is the representation of $G(\mathbb{R})^\circ$ corresponding to μ, $|\cdot|^{d/2}\rho_d^{\,\cdot}|H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty}\otimes\pi^\circ)|_{W_{\mathbb{C}}}$ is equvivalent to $\mathbb{C}_\mu$, and if E is real $|\cdot|^{d/2}\rho_d^{\,\cdot}|H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty}\otimes(\pi^\circ\oplus\bar\pi^\circ))$ is equvivalent to $\mathbb{C}_\mu\oplus\mathbb{C}_{\bar\mu}$ (if the Cartan decomposition of $\bar{g}_\infty$ determined by ad $h_o(i)$ is $\bar{k}_\infty\oplus\bar{p}$ then $H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty}\otimes\pi^\circ) = \operatorname{Hom}_{K_\infty}(\Lambda^d\bar{p}_{\mathbb{C}},\xi_{\pi_\infty}\otimes\pi^\circ)$ and this space is one-dimensional (BW, II, Theorem 5.3), of type (p,q) with $p = d/2 - \langle\mu,\delta_o\rangle$ and $q = d/2 + \langle\mu,\delta_o\rangle$ and the Hodge structure is given by $z\mapsto z^{-p'}\bar{z}^{-q'}$ with $p' = d/2 - \langle\mu,\Lambda_o\rangle$ and $q' = d/2 - \langle\mu,\iota'\Lambda_o\rangle$, if E is real ι^* maps $H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty}\otimes\pi^\circ)$ to $H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty}\otimes\bar\pi^\circ)$ and if $n\in G(\mathbb{R})$ represent $\bar\omega\in\mathcal{R}(G(\mathbb{R}),T_o(\mathbb{R}))$ ι^* is determined by the map on $\Lambda^d\bar{p}_{\mathbb{C}}$ given by ad n and the map $\xi_{\pi_\infty}\otimes\pi^\circ\to\xi_{\pi_\infty}\otimes\bar\pi^\circ$ given by $(\xi_{\pi_\infty}\otimes\pi)(n)$ (π is π° (or $\bar\pi^\circ$) induced to $G(\mathbb{R})$), this operator intertwines $\xi_{\pi_\infty}\otimes\pi^\circ$ and $(\xi_{\pi_\infty}\otimes\bar\pi^\circ)\circ$ (ad n).

We conclude that

$${}^{o}r\circ\varphi_\infty|_{W_{\mathbb{C}}} \sim |\cdot|^{d/2}\rho_d^{\,\cdot}|\bigoplus_{\pi\in\Pi_\infty} H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty}\otimes\pi)|_{W_{\mathbb{C}}}$$

and if E is real

$$r \circ \varphi_\infty \sim |\cdot|^{d/2} \rho_d^{\vee}\Big|\bigoplus_{\pi \in \Pi_\infty} H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty} \otimes \pi),$$

by inducing in the first case for E not real we have in both cases

$$r \circ \varphi_\infty \sim |\cdot|^{d/2}\mathrm{Ind}(W_{\mathbb{R}},W_{\mathbb{E}}, \rho_d^{\vee}\Big|\bigoplus_{\pi \in \Pi_\infty} H^d(\bar{g}_\infty,\bar{K}_\infty,\xi_{\pi_\infty} \otimes \pi)).$$

It is clear from the definition of the correspondance $\mathcal{R}\mu \leftrightarrow \{\pi^\bullet\}$ and of $r^{H,i}$ and Π_∞^{ih} that this equvivalence respect our decomposition when restricting to ${}^L H^o \rtimes W_{\mathbb{R}}$.

The formulas for ν not the chosen imbedding is now an immediate consequence of the fact that $(T_o)(\mathbb{R})$ is also a fundamental Cartan subgroup of ${}^\nu G(\mathbb{R})$ and that if $\pi^\bullet$ (as a representation of $G(\mathbb{R})^\bullet$) corresponds to $\mu \in \mathcal{R}\mu$ then $\pi^\bullet$ (as a representation of ${}^\nu G(\mathbb{R})^\bullet$) corresponds to $\tau\mu \in \mathcal{R}\tau\mu$ if ν is the chosen imbedding composed with $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.]

Almost all the statements in the following are conjectures - a reference is given if the conjecture is not a fabrication of mine.

Let Π be a L-packet of representations of $G(\mathbb{A})$, that is, Π is the restricted product over all places v of $\mathbb{Q}$ of L-packets Π_v of representations of $G(\mathbb{Q}_v)$, almost all Π_v are demanded to contain an unique representation which contains the trivial representation of K_v - we identify $\{\pi_v\} \in \Pi$ and $\bigotimes_v \pi_v$. Π is <u>automorphic</u> if some $\pi \in \Pi$ is automorphic. If some $\pi \in \Pi$ occurs (discretely) in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$ then the same is true for every automorphic $\pi \in \Pi$. Π is <u>cuspidal</u> if some $\pi \in \Pi$ is cuspidal, then every automorphic $\pi \in \Pi$ is cuspidal. Π is <u>isobaric</u> if $\Pi = \Pi(\varphi)$ for some $\varphi \in \Phi(G)$, then Π is automorphic (follows from the proposition of I4 if we have proved that $\Pi(\varphi)$ is cuspidal for φ elliptic). Π is <u>anomalous</u> if it is automorphic but not isobaric. For $G = GL(n)$, Π is always singleton (Bo) and Π is isobaric if it is cuspidal (conjecture B of L5 and the conjecture (also of L5) that a tempered L-packet is of the form $\Pi(\varphi)$, in fact, a cuspidal representation of $GL(n,\mathbb{A})$ is <u>per definition</u> isobaric in L5).

To every pair (M,Π^0)(up to conjugation by an element of $G(\mathbb{Q})$), where M is a $\mathbb{Q}$-Levy subgroup of G and Π^0 is a cuspidal L-packet of representations of $M(\mathbb{A})$, we can construct a set $\overline{\Pi}(M,\Pi^0)$ of automorphic L-packets of representations of $G(\mathbb{A})$: for each place v of $\mathbb{Q}$, the set $\{\pi \mid \exists \sigma \in \Pi_v^0 : \pi$ is a constituent of $\mathrm{Ind}(G(\mathbb{Q}_v), P(\mathbb{Q}_v), \sigma)$(P some $\mathbb{Q}$-parabolic subgroup of G containing M as a Levy subgroup)$\}$ is a finite union of L-packets of representations of $G(\mathbb{Q}_v)$, Π_v^0 lifted to $G(\mathbb{Q}_v)$(via ${}^LM_v \subset {}^LG_v$ *) and the principle of functoriality) is one of these L-packets (the inductive property of the (conjectural) Langlands correspondance) and this satisfies the above condition for almost all v, we can therefore form the restricted products of all combinations of these local L-packets - every such (global) L-packet is automorphic (proved in L4). Every automorphic L-packet belongs to $\overline{\Pi}(M,\Pi^0)$ for some (M,Π^0) (proved in L4) and the sets $\overline{\Pi}(M,\Pi^0)$ are disjoint (conjecture A of L5 for $G = GL(n)$). Π^0 lifted to $G(\mathbb{A})$ is a L-packet in $\overline{\Pi}(M,\Pi^0)$, it is denoted by $\Pi(M,\Pi^0)$. If Π is isobaric, then $\Pi = \Pi(M,\Pi^0)$, where if $\Pi = \Pi(\varphi)$, M is the Levy subgroup of G corresponding to the minimal relevant Levy subgroup ${}^LM$ of ${}^LG$ containing $\mathrm{Im}\,\varphi$, and $\Pi^0 = \Pi(\varphi_M)$ for $\varphi_M = \varphi$ regarded as mapping into ${}^LM$ (the definition of the principle of functoriality). For $G = GL(n)$, the isobaric L-packets are precisely those of the

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*) In the following we let ${}^LG$ denote ${}^LG^0{}_{*}\ldots$.

form $\Pi(M,\Pi^o)$ (because Π^o is always isobaric).

A L-packet Π is <u>tempered</u> if it is automorphic and each Π_v is tempered, then Π is isobaric (L5)(the corresponding φ is tempered and conversely) and the set $\overline{\Pi}(M,\Pi^o)$ is singleton $(=\{\Pi\})$(if an irreducible tempered representation of a (local) Levy subgroup is induced, the constituents should belong to the same L-packet).

If the automorphic L-packet Π is isobaric, say $\Pi = \Pi(\varphi)$ for $\varphi \in \Phi(G)$, we expect that the group $\mathcal{S}_\varphi = \pi_0(S_\varphi/Z)$ and the pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_\varphi \times \Pi \longrightarrow \mathbb{C}$ control the automorphic representations $\pi \in \Pi$: the multiplicity with which π occur in the space of automorphic forms is $d_\varphi |\mathcal{S}_\varphi|^{-1} \sum_{s \in \mathcal{S}_\varphi} \langle s, \pi \rangle$, here d_φ is the number of (global) equvivalence classes in the local equvivalence class containing φ. If Π is anomalous I guess that the automorphic representations in Π are controlled by a group of the same type: we can find a φ (belonging to $\Phi(G')$ for some inner form G' of G) such that $\Pi(\varphi)$ and Π are equal at almost all places and a pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_\varphi \times \Pi \longrightarrow \mathbb{C}$ having the above property.

According to the theory of Arthur (A1), the L-packets Π which "occur" in the regular representation of $G(\mathbb{A})$ should be parametrized by "admissible" homomorphisms $\bar{\varphi}: L_{\mathbb{Q}} \times SL_2(\mathbb{C}) \longrightarrow {}^LG$ in the same way as the isobaric Π are parametrized by admissible homomorphisms $\varphi: L_{\mathbb{Q}} \longrightarrow {}^LG$, however different Π can be associated to the same $\bar{\varphi}$, but these Π belong to the same set $\overline{\Pi}(M,\Pi^o)$: the $\bar{\varphi}$ parametrize some of these sets. The $\varphi \in \Phi(G')$ associated to Π is in this case given by $\varphi(w) = \bar{\varphi}(w, \mathrm{dia}(|w|^{\frac{1}{2}}, |w|^{-\frac{1}{2}}))$ and the association $\bar{\varphi} \mapsto \varphi$ is injective. If $\varphi \in \Phi(G)$, $\Pi(\varphi)$ is associated to $\bar{\varphi}$ (and is the isobaric

L-packet (that is $\Pi(M,\Pi^{0})$) in the set $\bar{\Pi}(M,\Pi^{0})$ associated to $\bar{\varphi}$, for $G = GL(n)$, $\Pi(\varphi)$ is the only L-packet associated to $\bar{\varphi}$). In the definition of admissibility it is required that $\bar{\varphi}|_{L_{\mathbb{Q}}}$ is essentially tempered (for $\bar{\varphi}|SL_2(\mathbb{C})$ trivial, $\Pi(\bar{\varphi}|_{L_{\mathbb{Q}}})$ is the (only) L-packet associated to $\bar{\varphi}$). We let $\bar{\bar{\Phi}}(G)$ denote the set (of equvivalence classes) of Arthur parametres.

There is a sign character $\epsilon_{\bar{\varphi}} : \mathcal{S}_{\bar{\varphi}} \to \{\pm 1\}$ and there should be a pairing $\langle\cdot,\cdot\rangle' : \mathcal{S}_{\bar{\varphi}} \times \Pi \to \mathbb{C}$ such that the multiplicity with which $\pi \in \Pi$ occur in the regular representation is $d_{\bar{\varphi}} |\mathcal{S}_{\bar{\varphi}}|^{-1} \sum_{s \in \mathcal{S}_{\bar{\varphi}}} \epsilon_{\bar{\varphi}}(s) \langle s,\pi\rangle'$. Π occurs discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$ iff $\bar{\varphi}$ is elliptic. We let $s_{\bar{\varphi}}$ denote $\bar{\varphi}(1\times(-1)) \in S_{\bar{\varphi}}$ and its image in $\mathcal{S}_{\bar{\varphi}}$. $S_{\bar{\varphi}}$ is a subgroup of S_{φ} and the homomorphism $\mathcal{S}_{\bar{\varphi}} \to \mathcal{S}_{\varphi}$ is surjective (and maps $s_{\bar{\varphi}}$ to 1). We define a new pairing $\langle\cdot,\cdot\rangle : \mathcal{S}_{\bar{\varphi}} \times \Pi \to \mathbb{C}$ by $\langle s,\pi\rangle = \frac{1}{2}(\epsilon_{\bar{\varphi}}(s)\langle s,\pi\rangle' + \epsilon_{\bar{\varphi}}(ss_{\bar{\varphi}}) \langle ss_{\bar{\varphi}},\pi\rangle')$, then the multiplicity formula reads $d_{\bar{\varphi}} |\mathcal{S}_{\bar{\varphi}}|^{-1} \sum_{s\in\mathcal{S}_{\bar{\varphi}}} \langle s,\pi\rangle = d_{\varphi} |\mathcal{S}_{\varphi}|^{-1} \sum_{s\in\mathcal{S}_{\varphi}} \langle s,\pi\rangle$ ($\langle\cdot,\pi\rangle$ should factorize through $\mathcal{S}_{\bar{\varphi}} \to \mathcal{S}_{\varphi}$).

If $\bar{\varphi} \in \bar{\bar{\Phi}}(G)$ and $s \in S_{\bar{\varphi}}$ we can (in the same way as in 1.14) construct an endoscopic datum $(H,\bar{s},\eta)$(up to isomorphism) and a $\bar{\psi} \in {}_{G}\bar{\bar{\Phi}}(H)$ such that $\eta(\bar{s}) = s$ and $\eta\circ\bar{\psi} \sim \bar{\varphi}$. This construction determines an equvivalence relation $\sim$ on $S_{\bar{\varphi}}$: $s \sim s' \Leftrightarrow$ the constructed $(H,\bar{s},\eta)$ and $\bar{\psi}$ are the same. We let $\mathcal{S}^{*}_{\bar{\varphi}} = S_{\bar{\varphi}}/\sim$, this set is finite and the projection $S_{\bar{\varphi}} \to \mathcal{S}_{\bar{\varphi}}$/conjugation should factorize through $S_{\bar{\varphi}} \to \mathcal{S}^{*}_{\bar{\varphi}}$, thus we have a projection $\mathcal{S}^{*}_{\bar{\varphi}} \to \mathcal{S}_{\bar{\varphi}}$/conjugation. The same construction applies to $\varphi \in \Phi(G)$, $s \in S_{\varphi}$.

If φ is associated to $\bar{\varphi}$, we have an injection $\mathcal{S}^{*}_{\bar{\varphi}} \to \mathcal{S}^{*}_{\varphi}$. If $\bar{\varphi} \in \bar{\bar{\Phi}}(G)_e$ (e = elliptic), $\mathcal{S}_{\bar{\varphi}} = S_{\bar{\varphi}}/Z$, and this group and $\mathcal{S}_{\varphi}$ are abelian. The image of $\mathcal{S}^{*}_{\bar{\varphi}} = \mathcal{S}_{\bar{\varphi}}$ in $\mathcal{S}^{*}_{\varphi}$ is denoted $(\mathcal{S}^{*}_{\varphi})_f$.

Proposition 11.3.2 of K3 (see 1.14) should remain true for $\bar{\varphi} \in \bar{\bar{\Phi}}(G)_e$ and also the (conjectural) considerations at the end of that paper: if $\bar{\varphi} \in \bar{\bar{\Phi}}(G)_e$, its contribution to the trace $\sum_{\Pi \sim \bar{\varphi}} \sum_{\pi \in \Pi} m_\pi \operatorname{tr} \pi(\phi)$ (ϕ a function on $G(\mathbb{A})$) can be stabelized as:

$$\sum_{(H,s,\eta)\in\mathcal{E}} \iota(G,H) \sum_{\Pi^H} \sum_{\pi\in\Pi^H} n_\pi \operatorname{tr} \pi(\phi^H),$$

here Π^H run over the automorphic L-packets of representations of $H(\mathbb{A})$ which lift to some Π associated to $\bar{\varphi}$, ϕ^H is a function on $H(\mathbb{A})$ connected with ϕ (see 3.7, in the formula there we must replace $\Phi(G)_{temp}$ by $\bar{\bar{\Phi}}(G)$, $\Phi(H)_{temp}$ by $\bar{\bar{\Phi}}(H)$, $\langle 1, \pi\rangle$ by $\langle s_{\bar{\varphi}}, \pi\rangle'$, $\langle \eta(s), \pi\rangle$ by $\langle \eta(s)s_{\bar{\varphi}}, \pi\rangle'$, and the summations must be taken over all Π^H resp. Π associated to $\bar{\varphi}$ resp. $\bar{\varphi}$) and $n_\pi = d_{\bar{\varphi}} |\mathcal{S}_{\bar{\varphi}}|^{-1} \epsilon_{\bar{\varphi}}(s_{\bar{\varphi}})$ $\langle s_{\bar{\varphi}}, \pi\rangle'$ is the stable multiplicity of π.

Now I can state the complete form of the expression for the zeta function in terms of L-functions:

$$\prod_{\substack{s\in(\mathcal{S}_\varphi^*)_f \\ \Pi}} \prod_{\varepsilon\in\{\pm 1\}\, i\in\mathcal{X}(s)} \Big(\prod\Big(\prod L(s-d/2,\psi_M, r_\varepsilon^{H,i})^{\sum_{\pi_f\in\Pi_f}\langle s,\, \pi_\infty^{i}\eta\cdot\xi\pi_f\rangle \operatorname{tr} \pi_f(\phi)}\Big)^{\varepsilon m(\pi_\infty^0) d_\varphi |(\mathcal{S}_\varphi^*)_f|^{-1}}\Big)^{*)} \qquad (**)$$

Here Π run over the L-packets of representations of $G(\mathbb{A})$ occuring (discretely) in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$ (and for which $Z(\mathbb{R})$ and Z_K acts as usual). φ ($\in \Phi(G'))$ is associated to Π as above. Let $\bar{\varphi} \in \bar{\bar{\Phi}}(G)_e$ be an Arthur parameter of Π, we can assume that $\bar{\varphi}_\infty(W_{\mathbb{C}}) \subset {}^LT^o \times W_{\mathbb{C}}$. The centralizer ${}^LM^o$ of $\bar{\varphi}_\infty(W_{\mathbb{C}})$ in ${}^LG^o$ is a Levy subgroup (containing ${}^LT^o$) and if $\bar{\varphi}_\infty(z) = z^{\Lambda}\bar{z}^{\iota\Lambda} \times z$ ($\Lambda \in X_*(Z_{{}^LM^o}) \otimes \mathbb{R}$), Λ determines a parabolic subgroup ${}^LP^o$ of ${}^LG^o$ with ${}^LM^o$ as Levy subgroup, $\bar{\varphi}_\infty(\tau)$ determines an action of $Gal(\mathbb{C}/\mathbb{R})$ on ${}^LM^o$. If

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*) Here and in some of the following formulas we should strictly speaking change the sign in $m(\pi_\infty^0)$ since we have defined the zeta function as the inverse product of the local zeta functions.

$\varphi'_M \in \bar{\Phi}(M)$ parametrizes the "trivial" discrete series representation of $M(\mathbb{R})$, then $\varphi^{\circ}_{\infty} : W_{\mathbb{R}} \xrightarrow{\varphi'_M} {}^{L}M^{\circ} \rtimes W_{\mathbb{R}} \xrightarrow{\mathrm{id} * \ldots} {}^{L}G^{\circ} \rtimes W_{\mathbb{R}}$ belongs to $\bar{\Phi}(G'_{\infty})$ (G'_{∞} quasi-split form of G_{∞}). We can restrict our attention to those $\bar{\varphi}$ for which $\bar{\varphi}_{\infty}$ and φ°_{∞} are elliptic (and $\bar{\varphi}(1, \{\begin{smallmatrix}1 & 1\\ & 1\end{smallmatrix}\})$ is regular unipotent in ${}^{L}M^{\circ}$), and we let $\Pi^{\circ}_{\infty} = \Pi(\varphi^{\circ}_{\infty})$. To φ and $s \in (\mathcal{S}^{*}_{\varphi})_f$ we construct a (H, s, η) and a $\psi \in \bar{\Phi}(H)$ as above. Define $\mu_{h_o} \in X^{*}({}^{L}T^{\circ})$ as in 1.9 ((H, s, η) need not be elliptic at infinity, but we can restrict our attention to those φ for which T_o can be chosen elliptic at infinity) and define μ_o $X^{*}({}^{L}T^{\circ})$ (from φ°_{∞}) as in 1.12, then $\eta = (\mu_o - \mu_{h_o})(s)$ and $\mathcal{H}(s) = \{(\mu - \mu_o)(s) | \mu \in \Omega \mu\}$ ($\subset$ roots of unity). A $\pi_{\infty} \in \Pi_{\infty}$ (for which $\langle s, \pi_{\infty} \rangle \neq$ o for some $s \in S_{\bar{\varphi}_{\infty}}$) is constructed from a Levy subgroup M of $G_{\mathbb{R}}$ and a parabolic subgroup P of $G_{\mathbb{C}}$ containing M as Levy subgroup. We can choose a fundamental Cartan subgroup T of $G_{\mathbb{R}}$ contained in M and a $h \in X_{\infty}$ factoring through T. The L-group of M is ${}^{L}M^{\circ} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, in this construction we have chosen an isomorphism $X_{*}(T) \leftrightarrow X^{*}({}^{L}T^{\circ})$ "transforming" P to ${}^{L}P^{\circ}$. To π_{∞} we associate the element $i' = (\mu_h - \mu_{h_o})(s) \in \mathcal{H}_h(s) = \{(\mu - \mu_{h_o})(s) | \mu \in \Omega_h \mu\}$ (this is well defined) and this association determines a disjoint family of subsets $\Pi^{i'}_{\infty} \subset \Pi_{\infty}$ ($\Pi^{i'}_{\infty}$ can be empty), we let $\Pi^{i',\varepsilon}_{\infty} = \{\pi \in \Pi^{i'}_{\infty} \mid \mu_h(s_{\bar{\varphi}}) = \varepsilon\}$. We have a bijection $\mathcal{H}(s) \longrightarrow \mathcal{H}_h(s)$ given by $i \mapsto i' = i\eta$. If ${}^{L}M$ resp. ${}^{L}M^{H}$ is the minimal Levy subgroup of ${}^{L}G$ resp. ${}^{L}H$ containing Im φ resp. Im ψ, then $s_{\bar{\varphi}} \in Z_{{}^{L}M}$ resp. $Z_{{}^{L}M^{H}}$, and $s_{\bar{\varphi}}$ determines a $\pm$-decomposition of $r|{}^{L}M$ resp. $r^{H,i}|{}^{L}M^{H}$.

The proof is an immediate generalizasion of step (1o)-(14) in section 2: In (1o) we shall replace $\Phi(H)_e$ by $\bar{\Phi}(H)_e$, $\Pi(\psi)$ by $\sum_{\Pi^{H} \sim \bar{\psi}}$ and $\langle 1, \pi \rangle$ by $\epsilon_{\bar{\varphi}}(s_{\bar{\varphi}}) \langle s_{\bar{\varphi}}, \pi \rangle$. $m(\Pi_{\omega})$ must be replaced by

$\sum_{\pi\sim\varphi_\infty}\sum_{\pi\in\Pi_\infty}\langle s_{\bar\varphi_\infty},\pi\rangle \operatorname{tr}\pi(f_\xi^G)$ and this should be equal to $\langle s_{\bar\varphi_\infty},\pi_\infty\rangle$ $\mu_h(s_{\bar\varphi_\infty})m(\Pi_\infty^G)$, where π_∞ (arbitrary) is associated to $\bar\varphi_\infty$. A similar change of $m(\Pi_\infty^H)$. We note the generalizations of 3.6 and 3.7. In (14) we shall incorporate $\epsilon_{\bar\varphi}(ss_{\bar\varphi})$ and $\sum_{\pi\sim\bar\varphi}$ and replace $\langle s,\pi\rangle$ by $\langle ss_{\bar\varphi},\pi\rangle$ (we use that $\epsilon_{\bar\varphi}(s_{\bar\varphi}) = \epsilon_{\bar\varphi}(ss_{\bar\varphi})$). We have a bijection $\{i\in\mathcal{H}(s)\mid r_{\mathcal{E}}^{H,i}\neq 0\}\longrightarrow\{i'\in\mathcal{H}(ss_{\bar\varphi})\mid r_{\mathcal{E}}^{H',i'}\neq 0\}$ given by $i\mapsto i'=\mathcal{E}\mu_0(s_{\bar\varphi})i$. Now the formula follows **from the fact** that $r_{\mathcal{E}}^{H,i}\cdot\varphi_M = r_{\mathcal{E}}^{H',i'}\cdot\varphi'_M$ and

$$\tfrac{1}{2}(\epsilon_{\bar\varphi}(ss_{\bar\varphi})\ \langle s,\Pi_\infty^{i\eta}\rangle\sum_{\pi_f\in\Pi_f}\langle ss_{\bar\varphi},\pi_f\rangle \operatorname{tr}\pi_f(\phi)\ +$$
$$\epsilon_{\bar\varphi}(s)\ \langle ss_{\bar\varphi},\Pi_\infty^{i'\eta'}\rangle\sum_{\pi_f\in\Pi_f}\langle s,\pi_f\rangle \operatorname{tr}\pi_f(\phi)) = \langle s_{\bar\varphi},\pi_\infty\rangle\sum_{\pi_f\in\Pi_f}\langle s,\Pi_\infty^{i\eta,\mathcal{E}}\otimes\pi_f\rangle \operatorname{tr}\pi_f(\phi),$$

where π_∞(arbitrary)$\in\Pi_\infty^{i\eta,\mathcal{E}}$. Of course, this "proof" works only locally at primes p satisfying our conditions in this paper.

Now we will compare this formula with a formula for the zeta function obtained **from a decomposition of the étal cohomology parametrized** by representations analogous to that of the rational cohomology used in our proof of (14) at the infinite place.

$G(\mathbb{A}_f)$ and so the Hecke algebra $\mathcal{H}(G(\mathbb{A}_f),K)$ (with coefficients in $\mathbb{Q}$ resp. $\mathbb{Q}_\ell$) acts on $H^i(S(K)(\mathbb{C}),F_\xi(K))$ and $H^i_{\acute{e}t}(S(K),\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})$ (if $g\in G(\mathbb{A}_f)$ and $K' = K\cap gKg^{-1}$ we have two morphisms $S(K')\longrightarrow S(K)$(defined over E) a) by right multiplication by g and b) by inclusion, these induce maps on cohomology:

$$H^i(S(K)(\mathbb{C}),F_\xi(K))\longrightarrow H^i(S(K')(\mathbb{C}),F_\xi(K'))\longrightarrow H^i(S(K)(\mathbb{C}),F_\xi(K))$$

and

$$H^i_{\acute{e}t}(S(K),\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})\longrightarrow H^i_{\acute{e}t}(S(K'),\mathcal{F}_\xi(K')_{\mathbb{Q}_\ell})\longrightarrow H^i_{\acute{e}t}(S(K),\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell}),$$

the left maps because the inverse image by a) of $F_\xi(K)$ resp.

$\mathcal{F}_{\xi}(K)$ is $F_{\xi}(K')$ resp. $\mathcal{F}_{\xi}(K')$, the right maps because we have a map from the direct image by b) of $F_{\xi}(K')$ resp. $\mathcal{F}_{\xi}(K')$ to $F_{\xi}(K)$ resp. $\mathcal{F}_{\xi}(K)$).

The actions of $\mathcal{H}(G(\mathbb{A}_f),K)_{\mathbb{Q}_\ell}$ and $\mathrm{Gal}(\bar{E}/E)$ on $H^i_{\acute{e}t}(S(K),\mathcal{F}_{\xi}(K)_{\mathbb{Q}_\ell})$ commute and lead to a decomposition

$$H^i_{\acute{e}t}(S(K),\mathcal{F}_{\xi}(K)_{\mathbb{Q}_\ell})\otimes \bar{\mathbb{Q}}_\ell = \bigoplus_{\pi} X^i(\pi_\infty)\otimes W(\pi_f)$$

(π as before), $X^i(\pi_\infty)$ is a $\mathrm{Gal}(\bar{E}/E)$-module and depends on π, $W(\pi_f)$ is an irreducible $\mathcal{H}(G(\mathbb{A}_f),K)_{\bar{\mathbb{Q}}_\ell}$-module. If we choose an imbedding $\bar{\mathbb{Q}}_\ell \to \mathbb{C}$ and tensorize both sides we get the former decomposition of $H^i(S(K)(\mathbb{C}),F_{\xi}(K))\otimes_{\mathbb{Q}}\mathbb{C}$ $(X^i(\pi_\infty)\otimes_{\bar{\mathbb{Q}}_\ell}\mathbb{C} = H^i(\mathfrak{g}_\infty,\tilde{K}_\infty,$ $\xi\otimes\pi_\infty)$ and $W(\pi_f)\otimes_{\bar{\mathbb{Q}}_\ell}\mathbb{C} = \pi_f^K)$, in fact, we have obtained the decomposition of $H^i_{\acute{e}t}(S(K),\mathcal{F}_{\xi}(K)_{\mathbb{Q}_\ell})\otimes\bar{\mathbb{Q}}_\ell$ by first decomposing into irreducible $\mathcal{H}(G(\mathbb{A}_f),K)_{\bar{\mathbb{Q}}_\ell}$-modules and then comparing with the decomposition of $H^i(S(K)(\mathbb{C}),F_{\xi}(K))\otimes_{\mathbb{Q}}\mathbb{C}$.

If the L-packet Π contributes to the above sum and $\pi_\infty\in\Pi_\infty$, **then if $\pi=\pi_\infty\otimes\pi_f$ contributes to the sum for some $\pi_f\in\Pi_f$,** we expect that the $\mathrm{Gal}(\bar{E}/E)$-module $X^i(\pi_\infty)$ is independent of the choise of π_f in Π_f, hence we can define the $\mathrm{Gal}(\bar{E}/E)$-module $X^i(\Pi_\infty) = \bigoplus_{\pi_\infty\in\Pi_\infty} X^i(\pi_\infty)$ (depending on Π).

For every finite place v of E we thus have (for ℓ chosen such that $v\nmid\ell$) a λ-adic representation $\rho'^{\,i}_v(\Pi_\infty)$ of W_{E_v} (via $W_{E_v}\to$ $\mathrm{Gal}(\bar{E}_v/E_v)$) on $X^i(\Pi_\infty)$ (of course $X^i(\Pi_\infty)$ **should be replaced by** a vector space over some finite extension of $\mathbb{Q}_\ell$), and for every infinite place v of E we have the former (complex) representation $\rho'^{\,i}_v(\Pi_\infty)$ of W_{E_v} on $X^i(\Pi_\infty)\otimes_{\bar{\mathbb{Q}}_\ell}\mathbb{C}$. By inducing we have a representation $\rho^i_v(\Pi_\infty)$ of $W_{\mathbb{Q}_{\bar{v}}}$ ($\bar{v}$ the place of $\mathbb{Q}$ divided by v).

This decomposition of the cohomology imply:

$$Z(s,S(K),\xi) = \prod_{\substack{\sigma \text{ pla-} \\ \text{ce of } E}} \prod_{\pi} \prod_{j} \Big(\prod_{\sigma} L(s,\rho_\sigma^j(\pi_\infty))^{\dim \pi_f^K}\Big)^{(-1)^j}$$

$$= \prod_{\Pi} \prod_{s\in(\mathcal{S}^*_\varphi)_f} \prod_{j} \Big(\prod_{\substack{i,\varepsilon \\ i\in\mathcal{H}_h(s) \\ \varepsilon=\pm 1}} \big(\prod_{\sigma} L(s,\rho_\sigma^j(\Pi_\infty^{i,\varepsilon}))\big)^a\Big)^{(-1)^j},$$

where $a = d_\varphi \, |(\mathcal{S}^*_\varphi)_f|^{-1} \sum_{\pi_f\in\Pi_f} \langle s, \Pi_\infty^{i,\varepsilon} \otimes \pi_f \rangle \operatorname{tr} \pi_f(\phi)$, and thus we should have *)

$$\prod_{\Pi\sim\bar\varphi} \prod_{\substack{j \\ (-1)^{d+j}=\varepsilon}} \Big(\prod_{\sigma} L(s,\rho_\sigma^j(\Pi_\infty))\Big) = L(s-d/2,\varphi_M, r_\varepsilon)^{|m(\Pi_\infty^0)|},$$

if $\langle 1,\pi\rangle \neq 0$ for some of the π associated to $\bar\varphi$ for which $\mu_h(s_{\bar\varphi}) = \varepsilon$ ($\pi_\infty \to \mu_h$), here both sides should decompose in accordance with i, that is, the subsets $\Pi_\infty^{i,\eta}$ of Π_∞ and the constituents $r^{H,i}$ of r restricted to ${}^L H$ – we expect that if π_∞ contributes to the cohomology of degree j, then $(-1)^{d+j} = \mu_h(s_{\bar\varphi})$. The λ-adic representation

$$\bigoplus_{\Pi\sim\bar\varphi} \bigoplus_{j} \rho^j(\Pi_\infty) \quad ((-1)^{d+j}=\varepsilon)$$

of $W_{\mathbb{Q}}$ (via $W_E \longrightarrow \operatorname{Gal}(\bar E/E)$ and inducing) should thus correspond (locally) to the complex representation

$$|m(\Pi_\infty^0)| \times |\cdot|^{-d/2}\, r_\varepsilon \circ \varphi_M$$

of $L_{\mathbb{Q}}$ (for this correspondance see Ta**) – at an infinite place $\rho^j(\Pi_\infty)$ must be defined as earlier and is actually complex).

If $\bar\varphi\in\bar\Phi(G)_e$ is such that an associated L-packet Π contributes to $H_{\text{ét}}(S(K),\mathcal{S}_\xi(K)_{\mathbb{Q}_\ell})\otimes\bar{\mathbb{Q}}_\ell$ for some $X_\infty (\subset \{\mathbb{S}\to G_{\mathbb{R}}\})$, K and ξ, the (λ-adic) representation $\bigoplus_{\Pi\sim\bar\varphi}\bigoplus_{j} \rho^j(\Pi_\infty)$ $((-1)^{d+j}=\varepsilon)$ of $W_{\mathbb{Q}}$ should be the $|m(\Pi_\infty^0)|$-fold of a representation $\rho_{\bar\varphi,X_\infty,\varepsilon}$ which should depend only on $\bar\varphi$, X_∞ and ε, but which should be independent of ξ and K, and this should correspond to the (complex) representation $|\cdot|^{-d/2} r_\varepsilon\circ\varphi_M$ of $L_{\mathbb{Q}}$ (in particular $\dim \rho_{\bar\varphi,X_\infty,\varepsilon} =$

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*) In order to deduce this we must of course incorporate a dependence of the Hecke algebra in the zeta function, see BL.

**) Strictly speaking a λ-adic representation must be replaced by its $\bar\Phi$-semisimplification.

dim r_{ε}) - we expect that $m(\pi_{\infty}^{o}) = (-1)^{d}$ * the multiplicity of the absolutely irreducible constituent of $\check{\xi}$ having the same infinitesimal (and central) character as π_{∞}^{o}. Otherwise stating: the (isobaric) representation of $GL(n,\mathbb{A})$ ($n=\dim \rho_{\bar{\varphi},X_{\infty},\varepsilon}$) corresponding to $\rho_{\bar{\varphi},X_{\infty},\varepsilon}$ (by the Langlands correspondance) should be $|\det|^{-d/2}$ * the representation of $GL(n,\mathbb{A})$ ($n=\dim r_{\varepsilon}$) obtained by lifting the (cuspidal) L-packet $\pi^{o} = \pi(\varphi_{M'})$ of representations of $M'(\mathbb{A})$ via $r_{\varepsilon}: {}^{L}M' \to GL(n,\mathbb{C})$, here ${}^{L}M'$ is the minimal (relevant w.r.t. G') Levy subgroup of ${}^{L}G$ containing Im φ and M' is the Levy subgroup of G' corresponding to ${}^{L}M'$ (this is proved in Ll for $G = GL(2)$ and φ cuspidal, but only locally for some types of φ_{p} - for this G the Shimura variety is not compact so a generalization of our theory is necessarily, see below, see also BL, HLR and Ra).

The point is now that the dependence of this representation of $GL(n,\mathbb{A})$ on the Shimura variety, that is on X_{∞}, should be reflexed only in r (which is constructed from X_{∞}), so that $\varphi \in \Phi(G')$ should be independent of S(K) and in fact should be the φ which we earlier have associated to π.

Since a L-function $L(s,\varphi,r)$ is known to converge absolutly for Re s sufficiently large, to extend meromorphic and to satisfy the functional equation $L(s,\varphi,r) = \varepsilon(s,\varphi,r)\ L(1-s,\check{\varphi},r)$ ($\check{\varphi}$ is the contragredient of φ, for the definition of $\varepsilon(s,\varphi,r)$ and for a proof see Ta), the zeta function (which we have regarded as a formal power series) should converge absolutly for Re s sufficiently large, extend meromorphic and satisfy a functional equation, in fact, this functional equation seems to have the expected form $Z(s,M) = \varepsilon(s,M)\ Z(1-s,\check{M})$ (M is a motive over an algebraic numberfield and $\check{M}$ is the dual motive, see Ta and D2):

$$Z(s,S(K),\xi) = \mathcal{E}(s,S(K),\xi)\, Z(1-s,S(K)(d),\check{\xi})$$

($M(d) = M \otimes T^{\otimes d}$ – T the Tate object, thus $Z(s,M(d)) = Z(s+d,M)$).
If $M(S(K),\xi)$ is the motive associated to $(S(K),\xi)$, the motive which we here associate to $(S(K),\xi)$ is of course $\bigoplus (-1)^{i+1} M^i(S(K),\xi)$. We should have $\check{M}^i(S(K),\xi) = M^{2d-i}(S(K)(d),\check{\xi}^i) = M^i(S(K)(i),\check{\xi})$, and so the homogeneous functional equation

$$Z^i(s,S(K),\xi) = \mathcal{E}^i(s,S(K),\xi)\, Z^i(i+1-s,S(K),\check{\xi}).$$

The functional equation follows from the fact that (the global) $\mathcal{E}(s,V)$ is additive and that we (by the above) have $\prod_{\pi \sim \varphi} \prod_i \mathcal{E}(s,\rho^i(\pi_\infty))^{(-1)^i}$ $((-1)^{d+i} = \mathcal{E}) = \mathcal{E}(s-d/2,\varphi_M,r_{\mathcal{E}})^{\mathcal{E} m(\pi)}$.

If S(K) is not proper (that is, G_{ad} is not anisotropic over $\mathbb{Q}$) we can still easy define a zeta function. But a definition which is appropriate for an expression of the zeta function in terms of L-functions require some preliminary work.

If S(K) has "good" reduction at $\mathfrak{p}$, that is, if $S_{\mathfrak{p}}(K)$ is defined and smooth, we have (by the Lefschetz fixed point formula)

$$\exp \sum_{j=1}^{\infty} \frac{|\omega_{\mathfrak{p}}|^{js}}{j} \left|S_{\mathfrak{p}}(K)(\kappa^j)\right| = \prod_{i=1}^{2d} \det(1 - |\omega_{\mathfrak{p}}|^s \Phi_{\mathfrak{p}} | H^i(S(K),\mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

The left hand side is clearly a $\mathbb{Q}$-rational function of "$|\omega_{\mathfrak{p}}|^s$", but if $S_{\mathfrak{p}}(K)$ is not proper we can not any more prove that the individual factors on the right have coefficients in $\mathbb{Q}$. This fact is however in reality inessential for us, for other reasons we have to choose another cohomology. In contrast to the compact case, the eigenvalues α of the Frobenius action on the cohomology (being algebraic since the ℓ-adic polynomial has algebraic coefficients) need no more be "pure", that is, satisfy $\log_p |\nu(\alpha)|^2 \in \mathbb{Z}$ for every infinite place ν of the solution field – this defect already appears for GL(2).

It seems as if the cohomology used to define the zeta function ought to satisfy this purity condition. Also we must demand that it have an appropriate decomposition parametrized by representations like that of the usual (ℓ-adic) cohomology for S(K) proper. The existence of a such cohomology would for instance allow us to prove the Ramanujan-Petersson conjecture for a L-packet π occuring discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$: if π_∞ is discrete (and almost all π_p have a Whittaker model), then almost all π_p are (essentially) tempered.

It is natural to choose a suitable compactification $\overline{S(K)}$ of S(K) and extend $\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell}$ to $\overline{S(K)}_{\overline{E}}$, and to study the image of the restriction map $H^i_{ét}(\overline{S(K)},\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell}) \longrightarrow H^i_{ét}(S(K),\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})$, or the image of the map $H^i_{ét,c}(S(K),\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell}) \longrightarrow H^i_{ét}(S(K),\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})$ (c = compact support). The first cohomology is clearly $\mathbb{Q}$-rational and pure, the second is pure but the $\mathbb{Q}$-rationallity is unknown. The Hecke algebra $\mathcal{H}(G(\mathbb{A}_f),K)$ acts semi-simply on both cohomology spaces, they therefore possess a decomposition into irreducible $\mathcal{H}_{\mathbb{Q}_\ell}$-modules, but this decomposition need not come from a decomposition parametrized by representations.

It seems as if the <u>intersection</u> <u>cohomology</u> $\mathbb{IH}^i(\overline{S(K)},\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})$ (references in BL and HLR) is the adequate cohomology for the definition of the zeta function: the purity seems present and is proved for S(K) proper, and it seems to have the correct decomposition property: we have $\mathbb{IH}^i(\overline{S(K)},\mathcal{F}_\xi(K)_{\mathbb{Q}_\ell})\otimes_{\mathbb{Q}_\ell}\mathbb{C} = \mathbb{IH}^i(\overline{S(K)},F_\xi(K))\otimes_{\mathbb{Q}}\mathbb{C}$, and the last space seems to be isomorphic to the L^2-cohomology space $H^i_{(2)}(S(K),F_{\xi}(K)_{\mathbb{C}})$ (the conjecture of Zucker), but the L^2-cohomology is isomorphic to the $\mathfrak{g}$-$\mathfrak{k}$-cohomology, and this seems also in the non-compact case to possess the decomposition parametrized by representations occuring discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$.

In L1, BL and HLR the cases of Hilbert-Blumenthal varieties are treated ($G = \mathrm{Res}_{F/\mathbb{Q}}GL(2)$, F real numberfield). To define the intersection cohomology we let $\overline{S(K)}$ be the Satake-Baily-Borel compactification. This is <u>not</u> smooth. $S(K)$ and $\overline{S(K)}$ are defined over $\mathbb{Q}$ and the frontier $S(K)_{\infty} = \overline{S(K)}\backslash S(K)$ is finite. If $K = K_n$, $S(K)$ is defined over spec($\mathbb{Z}$ [1/n]) and there is an open subset W of spec ($\mathbb{Z}$ [1/n]) such that S(K) restricted to W has a compactification which after base-change by spec($\mathbb{Q}$) $\to$ W become $\overline{S(K)}$, also we can construct a <u>smooth</u> compactification of S(K) over spec($\mathbb{Z}$ [1/n]) which over W is a resolution of singularities of $\overline{S(K)}$.

The expression (**) for the zeta function in terms of L-functions should hold also in the non-compact case. In the proof a non-elliptic part of the trace comes into play, that is, a part origin from other parabolic subgroups of G than G. This part is the contribution to the sum (1) from the frontier $S_{\mathfrak{p}}(K)_{\infty}$. For the above Shimura varieties this contribution is only non-zero for $F = \mathbb{Q}$, the case studied in L1.

All the existing proofs of (special (multidimensional) cases of) formula (**) - where S(K) thus may be non-compact and where the reduction at p may be bad - have a look like our proof in this paper. It is always assumed that $S_{\mathfrak{p}}(K)$ exist for $\mathfrak{p}|p$ and that K_p is maximal compact. A such proof will however in this generality, strictly speaking, lead to an expression for the <u>semi-simple</u> zeta function in terms of <u>semi-simple</u> L-functions (precisely: $L^{ss}(s-d/2,\bar{\varphi}_M, r_{\mathcal{E}}^{H,i}) - \varphi_M$ must be replaced by $\bar{\varphi}_M$).

The generalization of our proof can be outlined in the following way. We have a diagram:

$$\begin{tikzcd}
\overline{S_{\mathfrak{p}}(K)}_{\bar{E}_{\mathfrak{p}}} \arrow[r, "j"] \arrow[d] & \overline{S_{\mathfrak{p}}(K)} \arrow[d] & \overline{S_{\mathfrak{p}}(K)}_{k} \arrow[l, "i"'] \arrow[d] \\
\mathrm{spec}(\bar{E}_{\mathfrak{p}}) \arrow[r] & \mathrm{spec}(O_{E_{\mathfrak{p}}}) & \mathrm{spec}(k). \arrow[l]
\end{tikzcd}$$

Let $IC^{\bullet}(\overline{S_{\mathfrak{p}}(K)}_{\bar{E}_{\mathfrak{p}}}, \mathcal{S}_{\xi}(K)_{\mathbb{Q}_{\ell}})$ be the cochain-complex used to define the intersection cohomology, and let, for $x \in \overline{S_{\mathfrak{p}}(K)}(k^j)$, $Tr_{x,j}$ be the <u>alternating</u> trace of the action of the Frobenius over k^j on the inertia invariants in the sheaves $i^{*}\mathcal{H}(j_{*}IC^{\bullet}(\overline{S_{\mathfrak{p}}(K)}_{\bar{E}_{\mathfrak{p}}}, \mathcal{S}_{\xi}(K)_{\mathbb{Q}_{\ell}}))$ on $\overline{S_{\mathfrak{p}}(K)}_{k}$ at the point x (the sheaves of <u>vanishing</u> <u>cycles</u> – $Gal(\bar{E}_{\mathfrak{p}}/E_{\mathfrak{p}})$ acts on these sheaves)*). In formula (1) we must replace $tr(\Phi_{\mathfrak{p}}^{j})_x$ by $Tr_{x,j}$, and sum over $\overline{S_{\mathfrak{p}}(K)}(k^j)$. If we ignore the contribution from the frontier $S_{\mathfrak{p}}(K)_{\infty}$ to the zeta function (or assume that this is zero, that is, $Tr_{x,j} = 0$ for $x \in S_{\mathfrak{p}}(K)_{\infty}(k^j)$, cf. the above remark), we can in formula (2) be content with replacing $|(I_{\emptyset})_{\mathcal{E}}\backslash(Y_p^j \times Y^p)|$ by $\sum_{x \in \mathcal{A}(\emptyset,\mathcal{E})(k^j)} \widetilde{Tr}^{o}_{x,j}$, here Tr^{o} is Tr for ξ trivial and $\mathcal{A}$ is defined on p. 51. $f_{\mathfrak{p},n}$ in formula (3) have to be defined in terms of $Tr^{o}_{x,j}$ (see Ra). In order to get (12) we shall use that tr $\pi_p(f^{H}_{\mathfrak{p},j}) = (1/j)|\omega^j|^{-d/2} \sum_{i \in \mathcal{X}} i \times$ [the <u>semi-simple</u> trace of the action of the j-th power of a Frobenius on the space of the ℓ-adic representation associated to the representation $\tilde{r}^{H,i}_{\mathfrak{p},j} \circ \bar{\varphi}_p : L_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \to GL(V_r^i)$].

If we assume that "the monodromy filtration of $IH^{\bullet}(\overline{S_{\mathfrak{p}}(K)}_{\bar{E}_{\mathfrak{p}}}, \mathbb{Q}_{\ell})$ is pure" (a conjecture of Deligne, see Ra), then the proved expression for the semi-simple zeta function in terms of semi-simple L-functions should imply our wanted formula (**).

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*) It is conjectured that the inertia group acts through a finite factor group.