3. List of conjectures.

3.1 If E is unramified over p, if G is quasi-split over $\mathbb{Q}_p$, if K_p is hyperspecial and if K^p is so small that $S(K)$ has good reduction modulo the prime ideal $\mathfrak{p}$ of E over p (that is, the reduced variety $S_{\mathfrak{p}}(K)$ exist and is proper and smooth), then the set of equvivalence classes of permissible homomorphisms $\phi: \mathcal{Q} \to \mathcal{G}$ can be put into a bijective correspondance with a class decomposition of $S_{\mathfrak{p}}(K)(\bar{k})$ in which each class is invariant under the Frobenius action, and the class corresponding to ϕ can be put into a bijective correspondance with $X_{\phi}(K)$ such that the action of the Frobenius on the class corresponds to the action of Φ on $X_{\phi}(K)$.

The proof of this conjecture seems to be the most difficult part of the theory, and I will sketch the proof in some of the cases in which the Shimura variety $S(K)$ parametrizes a family of polarized abelian varieties with endomorphism- and level structure (of type K). G is the group of symplectic similitudes on a $\mathbb{Q}$-vector space V w.r.t. a non-degenerate alternating bilinear form ψ (on V) and the action (on V) of a simple $\mathbb{Q}$-algebra D of degree d^2 over its center L, that is, $G = \{g \in GL_D(V) \mid \psi(gu,gv) = \psi(c(g)u,v),\ c(g) \in L_o\}$, D is endowed with a positive involution $*$, ψ satisfies $\psi(xu,v) = \psi(u,x^*v)$ $(x \in D)$ and L_o is the fixed field of $*$ on L. There exist a homomorphism $h: \mathbb{S} \to G_{\mathbb{R}}$ defined over $\mathbb{R}$ such that the corresponding Hodge structure on $V \otimes \mathbb{R}$ is of type $(1,o)+(o,1)$ and such that $\psi(u,h(i)v)$ is symmetric and positive definite. We choose an order O_D of D and an O_D-invariant lattice $V_{\mathbb{Z}}$ of V, and we choose p such that p is unramified in D, $D \otimes \mathbb{Q}_p$ is a product of matrix algebras, $O_D \otimes \mathbb{Z}_p$ is a maximal order and $\psi: V_{\mathbb{Z}_p} \times V_{\mathbb{Z}_p} \longrightarrow \mathbb{Z}_p$ is perfect, then $*(O_D \otimes \mathbb{Z}_p) = O_D \otimes \mathbb{Z}_p$ and we take $K_p = G(\mathbb{Q}_p) \cap \mathrm{End}_{O_D} V_{\mathbb{Z}_p}$. If K^p is sufficiently small then the pair (G,h) and $K = K_p \cdot K^p$ def- Shimura variety $S(K)$ satisfying all our wanted properties. The definition field E of $S(K)$ is the subfield of $\bar{\mathbb{Q}}$ generated by the image of the linear map $t: D \longrightarrow \bar{\mathbb{Q}}$ given by $t(x) = \mathrm{tr}(x \mid V_h^{1,0})$.

$S(K)(\mathbb{C})$ can be put into a bijective correspondance with the set of (isomorphy classes of) quadruples $(A,\iota,\Lambda,\bar{\eta})$, where A is an abelian variety over $\mathbb{C}$ up to isogeny, ι is a homomorphism $D \longrightarrow \mathrm{End}\, A$ such that $\mathrm{tr}(x \mid \mathrm{Lie}^* A) = t(x)$ for $x \in D$ ($\mathrm{Lie}^* A$ is the cotangent space of A), Λ is a L_0-homogeneous polarization on A which induces the involution $*$ on D and $\bar{\eta}$ is an equvivalence class for the action of K of $D \otimes \mathbb{A}_f$-module isomorphisms $\eta: H^1(A,\mathbb{A}_f) \xrightarrow{\sim} V \otimes \mathbb{A}_f$ which transform ψ to the form on $H^1(A,\mathbb{A}_f)$ induced by a polarization in Λ up to multiplication by an element of $L_0 \otimes \mathbb{A}_f$.

$S_{\mathfrak{p}}(K)(\bar{k})$ can be put into a bijective correspondance with the set of (isomorphy classes of) quadruples $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\tilde{\eta}})$, where $\tilde{A}$ is an abelian variety over $\bar{k}$ up to isogeny <u>of degree prime to p</u>, $\tilde{\iota}$ is a homomorphism $O_D \longrightarrow \mathrm{End}\,\tilde{A}$ such that $\mathrm{tr}(x \mid \mathrm{Lie}^* \tilde{A}) = t(x)$ for $x \in O_D$, $\tilde{\Lambda}$ is a L_0-homogeneous polarization on $\tilde{A}$ which induces the involution $*$ on O_D and which contains a polarization of degree prime to p, and $\bar{\tilde{\eta}}$ is an equvivalence class for the action of K^p of $O_D \otimes \mathbb{A}_f^p$-module isomorphisms $\tilde{\eta}: H^1(\tilde{A},\mathbb{A}_f^p) \xrightarrow{\sim} V \otimes \mathbb{A}_f^p$ which transform ψ to the form on $H^1(\tilde{A},\mathbb{A}_f^p)$ induced by a polarization in $\tilde{\Lambda}$ up to multiplication by an element of $L_0 \otimes \mathbb{A}_f^p$. An isogeny from $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\tilde{\eta}})$ to $(\tilde{A}',\tilde{\iota}',\tilde{\Lambda}',\bar{\tilde{\eta}}')$ is an isogeny from $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ to $(\tilde{A}',\tilde{\iota}',\tilde{\Lambda}')$, an isogeny of degree prime to p is an isomorphism. The class decomposition of $S_{\mathfrak{p}}(K)(\bar{k})$ is in our special case the isogeny classes.

The proof falls into two parts. In the first part it is proved that the set of equvivalence classes of permissible homomorphisms $\phi : \mathfrak{Q} \to \mathfrak{G}$ parametrize the set of isogeny classes in $S_{\mathfrak{p}}(K)(\bar{k})$. In the second part it is proved that an isogeny class has the decripted structure. The first part will be presented in two variants. The first builts on some unproved conjectures from the algebraic geometry, the second do not need any unproved conjectures but instead a theorem of Kottwitz (which was unproved at the time LR was publishing but which is now proved (by Kottwitz (unpublished) and (independently) by Reimann and Zink (RZ))).

The first variant can be outlined in the following way:

By using the Grothendieck standard conjectures we can construct the Tannakian category $M_{\bar{k}}$ (over $\mathbb{Q}$) of (all) motives over $\bar{k}$. We can (without use of unproved results) construct the neutral Tannakian category $M_{\bar{\mathbb{Q}}}$ (over $\mathbb{Q}$) of (all) motives over $\bar{\mathbb{Q}}$, the associated affine $\mathbb{Q}$-group is the connected motivic Galois group G^o (we have chosen an imbedding $\bar{\mathbb{Q}} \to \mathbb{C}$). A sub-Tannakian category $CM_{\bar{\mathbb{Q}}}$ of $M_{\bar{\mathbb{Q}}}$ is generated by the abelian varieties over $\mathbb{C}$ with complex multiplication and the Tate object, the associated affine $\mathbb{Q}$-group is the connected Serre group S. We therefore have a projection $G^o \to S$. Any abelian variety over $\mathbb{C}$ with complex multiplication can be reduced modulo p (we have chosen an imbedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ determining $\mathfrak{p}$) and the reduced variety determines a motive in $M_{\bar{k}}$. By using the Hodge conjecture for abelian varieties over $\mathbb{C}$ with complex multiplication we can extend this operation to a functor $CM_{\bar{\mathbb{Q}}} \longrightarrow M_{\bar{k}}$. If $L \subset \bar{\mathbb{Q}}$ is a CM-field and ${}^{L}CM_{\bar{\mathbb{Q}}}$ is the sub-Tannakian category of $CM_{\bar{\mathbb{Q}}}$ generated by the abelian varieties over $\mathbb{C}$ with complex multiplication through L and the Tate object, then the associated affine $\mathbb{Q}$-group is ${}^{L}S$, and if we let ${}^{L}M_{\bar{k}}$ denote the sub-Tannakian category of $M_{\bar{k}}$ generated by the image of ${}^{L}CM_{\bar{\mathbb{Q}}}$ by the reduction functor, then ${}^{L}M_{\bar{k}}$ is algebraic and by using the <u>Tate conjecture over a finite field</u> we can prove that "the" gerb associated to ${}^{L}M_{\bar{k}}$ is $\mathfrak{P}^{L}$(constructed in LR and in the appendix, we have a homomorphism $\mathfrak{Q}\to\mathfrak{P}$). We therefore have an injective homomorphism of gerbs $\mathfrak{P}^{L}\to\mathfrak{G}_{L_S}$ (determined up to conjugation by an element of $P^{L}(\bar{\mathbb{Q}})$).

Now let $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\bar{\eta}})$ be a point of $S_{\mathfrak{p}}(K)(\bar{k})$. To $\tilde{A}$ is associated a motive in $M_{\bar{k}}$ (belonging to ${}^{L}M_{\bar{k}}$ for L sufficiently large), the homogene part of degree 1 of this motive corresponds to a representation of $\mathfrak{P}$. We can assume that the representation space is V, that the action of D on V determined by $\tilde{\iota}$ is the given action and that some polarization $\tilde{\lambda}\in\tilde{\Lambda}$ corresponds to ψ on V. Then the representation maps into $\mathfrak{G}$ and the composition of this homomorphism $\mathfrak{P}\to\mathfrak{G}$ with $\mathfrak{Q}\to\mathfrak{P}$ is a permissible homomorphism $\phi:\mathfrak{Q}\to\mathfrak{G}$ (that ϕ is permissible is easy seen in the setting of the second variant of the proof below). If we had chosen another $\tilde{\lambda}\in\tilde{\Lambda}$ then the new ϕ would be equvivalent to the former, and if $(\tilde{A}',\tilde{\iota}',\tilde{\Lambda}',\bar{\bar{\eta}}')$ is isogene to $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\bar{\eta}})$ then the corresponding equvivalence class of permissible homomorphisms $\mathfrak{Q}\to\mathfrak{G}$ is the same. Conversely: a permissible homomorphism $\phi:\mathfrak{Q}\to\mathfrak{G}$ factorizes through $\mathfrak{Q}\to\mathfrak{P}$ and gives thus rise to a representation of $\mathfrak{P}$ and so a motive in $M_{\bar{k}}$, this motive is the homogene part of degree 1 of the motive associated to an abelian variety $\tilde{A}$ over $\bar{k}$, the action of D on the representation space V of ϕ determines an action $\tilde{\iota}$ of O_D on $\tilde{A}$, and the form ψ on V determines a L_o-homogeneous polarization $\tilde{\Lambda}$ on $\tilde{A}$, finally there exist a level structure $\bar{\bar{\eta}}$ on $\tilde{A}$ (because ϕ is permissible). Thus we have constructed a point $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\bar{\eta}})$ of $S_{\mathfrak{p}}(K)(\bar{k})$, another choise of $\emptyset$ (equvivalent to the former) would lead to an isogene point of $S_{\mathfrak{p}}(K)(\bar{k})$. These two maps between the set of equvivalence classes of permissible homomorphisms $\emptyset : \mathcal{Q} \longrightarrow \mathcal{G}$ and the set of isogeny classes of $S_{\mathfrak{p}}(K)(\bar{k})$ are the inverse of each other.

Then we come to the second variant.

A special point of $S(K)(\mathbb{C})$ is a triple (T,h,g), where T is a Cartan subgroup of G, $h \in X_{\infty}$ and factorizes through T and $g \in G(\mathbb{A}_f)$ (two triples are equvivalent (and identifies) if they differ by action of $G(\mathbb{Q})$ on the left and action of K on the right of g). In the above correspondance between points of $S(K)(\mathbb{C})$ and abelian varieties with additional structures, a special point corresponds to a sixtubel $(A,\iota,\Lambda,\bar{\eta},R,\vartheta)$(up to isomorphism), where the quadrupel $(A,\iota,\Lambda,\bar{\eta})$ corresponds to the point $\{(h,g)\}$ and R is the CM-algebra (= product of CM-fields) defining T (thus $T(\mathbb{Q}) = \{ r \in R^{*} \mid r \cdot \bar{r} \in L_o \}$ and $\dim_{\mathbb{Q}} R = \dim_{\mathbb{Q}} V/d$) and ϑ is a complex multiplication through R on (A,ι,Λ) (that is, an involution preversing imbedding $R \longrightarrow \mathrm{End}_D A$). This sixtubel can be constructed as follows: $\mu_h : \mathbb{C}^{*} \longrightarrow (R \otimes \mathbb{C})^{*}$ determines a complex multiplication (R,Φ), if B is the (polarizable) abelian variety over $\mathbb{C}$ up to isogeny with complex multiplication (R,Φ), we take $A = B^d$. Because $D \otimes_L R = M_d(R)$, D acts on A, this is ι. The representation space of the representation of ${}^K S$ (K sufficiently large field) corresponding to A (or rather, to the homogeneous part of degree 1) can be identified with V such that the action of D defined by ι is the given action and the "diagonal" action of R on V is that of T. We let Λ be the L_o-homogeneous polarization on A defined by ψ, and $\bar{\eta}$ be the set of isomorphisms $H^1(A,\mathbb{A}_f) = V \otimes \mathbb{A}_f \xrightarrow{\sim} V \otimes \mathbb{A}_f$ given by Kg^{-1} and we let ϑ be the "diagonal" action of R on A. If we reduce $(A,\iota,\Lambda,\bar{\eta},R,\vartheta)$ modulo p we get a special point $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\tilde{\eta}},R,\tilde{\vartheta})$ of $S_{\mathfrak{p}}(K)(\bar{k})$.

The second variant can be outlined in the following way:

Given (T,h), if we choose a $g \in G(\mathbb{A}_f)$, then to (T,h,g) we have constructed a special point $(A,\iota,\Lambda,\bar{\eta},R,\vartheta)$ of $S(K)(\mathbb{C})$ and (by reduction modulo p) a special point $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\tilde{\eta}},R,\tilde{\vartheta})$ of $S_{\mathfrak{p}}(K)(\bar{k})$, the isogeny class of $S_{\mathfrak{p}}(K)(\bar{k})$ containing the point $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\tilde{\eta}})$ is independent of the choise of g. The isogeny classes of $S_{\mathfrak{p}}(K)(\bar{k})$ constructed from (T,h) and (T',h') are equal if and only if ψ_{T,μ_h} and $\psi_{T',\mu_{h'}}$ (see appendix) are equvivalent. This is a consequence of the fact that the existence of an isogeny from $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ to $(\tilde{A}',\tilde{\iota}',\tilde{\Lambda}')$ is equvivalent to the existence of an automorphism g of $V \otimes \bar{\mathbb{Q}}$ satisfying the conditions (we have here identified $H^1(A,\mathbb{Q})$ and V in such a way that ι corresponds to the given action of D on V and that the bilinear form ψ_λ on $H^1(A,\mathbb{Q})$ associated to some $\lambda \in \Lambda$ corresponds to ψ, and analogous for A'):

1) g commutes with the action of D
2) g transforms Λ' to Λ
3) if we identify the contravariant rational Dieudonné module associated to $\tilde{A}$ resp. $\tilde{A}'$ with $V \otimes k$, where the F-translation is given by $x \mapsto \tilde{b}\sigma(x)$ resp. $x \mapsto \tilde{b}'\sigma(x)$, with $\tilde{b} = \chi(\tilde{b}_o)$ resp. $\tilde{b}' = \chi'(\tilde{b}_o)$ for $\tilde{b}_o \in {}^K S(k)$, then we can choose $s \in T(\bar{\mathbb{Q}}_p)$ such that $\bar{g} = gs \in G(\mathbb{Q}_p^{un})$ and $\tilde{b}' = \bar{g}\tilde{b}\sigma(\bar{g})^{-1}$ (for χ and χ' see below)
4) if we identify the ℓ-adic ($\ell \neq p$) cohomology spaces associated to $\tilde{A}$ and $\tilde{A}'$ with inner forms of $V \otimes \mathbb{Q}_\ell$, then g shall transform these spaces to each other
5) if the Frobenius endomorphisms on $\tilde{A}$ and $\tilde{A}'$ over k^j (for j

sufficiently large) correspond to the automorphisms $\tilde{\varepsilon}$ and $\tilde{\varepsilon}'$ on V, then we shall have $\tilde{\varepsilon}' = g\tilde{\varepsilon}g^{-1}$ (for j sufficiently large),

these conditions for g are equvivalent to the conditions:

1') $g \in G(\bar{\mathbb{Q}})$

2') g is an equvivalence for the two homomorphisms (*) on the kernel

$$\mathcal{P} \xrightarrow{\psi_{\mu_0}} \mathcal{G}_{K_S} \underset{\chi'}{\overset{\chi}{\rightrightarrows}} \begin{matrix}\mathcal{G}_T\\ \mathcal{G}_{T'}\end{matrix} \subset \mathcal{G} \qquad (*)$$

here μ_0 is the canonical cocharacter of ${}^K S$, ψ_{μ_0} is defined in the appendix and the homomorphisms $\chi: {}^K S \to T$ and $\chi': {}^K S \to T'$ are defined over $\mathbb{Q}$ and map μ_0 to μ_h and $\mu_{h'}$

3') g is a locally equvivalence for the two homomorphisms (*) w.r.t. $\zeta_\infty: \mathcal{W} \to \mathcal{P}$, $\zeta_p: \mathcal{D} \to \mathcal{P}$ and $\zeta_\ell: \mathcal{G}_\ell \to \mathcal{P}$ (for $\ell \neq p$).

[Sketch of proof: 2') follows from 5) and the definition of ψ_μ. 2) is tantamount to $\psi(gx,gy) = \psi(ax,y)$ for some $a \in L_0 \otimes \bar{\mathbb{Q}}$, and $h'(i) * \iota = g(h(i) * \iota)g^{-1}$, but since $v(h_0(i) * \iota)v^{-1} = \mu_0(-1) * \iota$ $= (\psi_{\mu_0} \circ \zeta_\infty)(\tau)$, where $v =$ (say) $(\mu_0 + \bar{\mu}_0)(\sqrt{i})$, we have $\psi_{\mu_{h'}} \circ \zeta_\infty$ $= \mathrm{ad}\, g \circ (\psi_{\mu_h} \circ \zeta_\infty)$. If $\tilde{b}_0 \in {}^K S(k)$ determines the F-translation and $b_0 \in {}^K S(k)$ is constructed from $\psi_{\mu_0} \circ \zeta_p : \mathcal{D} \to \mathcal{G}_{K_S}$ (as in 1.2), then the <u>theorem of Kottwitz</u> states that $b_0 = u_0 \tilde{b}_0 \sigma(u_0)^{-1}$ for some $u_0 \in {}^K S(k)$, in fact $u_0 \in \mathrm{Im}\, \psi_{\mu_0}(P(k))$, we therefore have $b =$ $u\tilde{b}\sigma(u)^{-1}$, $b' = u'\tilde{b}'\sigma(u')^{-1}$ and $u' = \bar{g}u\bar{g}^{-1}$ ($u = \chi(u_0)$, ...), the condition $\tilde{b}' = \bar{g}\tilde{b}\sigma(\bar{g})^{-1}$ is then equvivalent to $b' = \bar{g}b\sigma(\bar{g})^{-1}$, this implies that b' also can be constructed from $\mathrm{ad}\, g \circ (\psi_{\mu_h} \circ \zeta_p)$, therefore we must have $\psi_{\mu_{h'}} \circ \zeta_p = \mathrm{ad}\, g \circ (\psi_{\mu_h} \circ \zeta_p)$. The above mentioned forms of $V \otimes \mathbb{Q}_\ell$ are determined by a homomorphism ζ'_ℓ : $\mathcal{G}_\ell \to \mathcal{P}$ (a trivialization) and this is equvivalent to ζ_ℓ:

$\mathcal{G}_\ell \to \mathcal{P}$, 4) is tantamount to $\psi_{\mu_{h'}} \cdot \mathcal{S}_\ell' = \mathrm{ad}\ g \cdot (\psi_{\mu_h} \cdot \mathcal{S}_\ell')$, but this condition is equvivalent to $\psi_{\mu_{h'}} \cdot \mathcal{S}_\ell = \mathrm{ad}\ g \cdot (\psi_{\mu_h} \cdot \mathcal{S}_\ell)$ (because $\psi_{\mu_{h'}} \cdot \mathcal{S}_\ell = \mathrm{ad}\ y' \cdot (\psi_{\mu_{h'}} \cdot \mathcal{S}_\ell') = \mathrm{ad}\ y' \cdot \mathrm{ad}\ g \cdot (\psi_{\mu_h} \cdot \mathcal{S}_\ell') = \mathrm{ad}\ g \cdot \mathrm{ad}\ y \cdot (\psi_{\mu_h} \cdot \mathcal{S}_\ell') = \mathrm{ad}\ g \cdot (\psi_{\mu_h} \cdot \mathcal{S}_\ell)$, here $y = \psi_{\mu_h}(\gamma)$ and $y' = \psi_{\mu_{h'}}(\gamma)$ if $\mathcal{S}_\ell = \mathrm{ad}\,\gamma \cdot \mathcal{S}_\ell'$ for $\gamma \in P(\bar{\mathbb{Q}}_\ell)$).]

Now we shall use that two homomorphisms $\psi, \psi' : \mathcal{P} \to \mathcal{G}$ are equal if they are equal on the kernel and locally equal and that the two homomorphisms (*) composed with the homomorphism $\mathcal{Q} \to \mathcal{P}$ are ψ_{T,μ_h} and $\psi_{T',\mu_{h'}}$.

Since every permissible homomorphism $\phi : \mathcal{Q} \to \mathcal{G}$ is equvivalent to one of the form ψ_{T,μ_h} (LR, Satz 5.3) we can consequently define an injective map from the set of equvivalence classes of permissible homomorphisms $\phi : \mathcal{Q} \to \mathcal{G}$ to the set of isogeny classes of $S_{\mathfrak{p}}(K)(\bar{k})$. This map is surjective because every point $(\tilde{A}, \tilde{\iota}, \tilde{\Lambda}, \bar{\tilde{\eta}})$ of $S_{\mathfrak{p}}(K)(\bar{k})$ is component of a special point $(\tilde{A}, \tilde{\iota}, \tilde{\Lambda}, \bar{\tilde{\eta}}, R, \tilde{\mathcal{J}})$ for some R and $\tilde{\mathcal{J}}$ (because $\tilde{A}$ is defined over a finite field), and a special point of $S_{\mathfrak{p}}(K)(\bar{k})$ is the reduction modulo p of a special point of $S(K)(\mathbb{C})$(Z2, § 4.4).

Now we come to the second part of the proof.

Let $\phi : \mathcal{Q} \to \mathcal{G}$ be a permissible homomorphism, and let $\mathcal{A} \subset S_{\mathfrak{p}}(K)(\bar{k})$ be the corresponding isogeny class, then we shall construct a bijection $\mathcal{A} \xrightarrow{\sim} I_\phi \backslash (X_p \times X^p/K^p)$ such that the Frobenius action (over k) on $\mathcal{A}$ corresponds to the action of $\Phi = (b \times \sigma)^r$ on X_p. We can assume that $\phi = \psi_{T,\mu_h}$ and we choose a $g \in G(\mathbb{A}_f)$. To (T,h,g) we have constructed a special point $(A, \iota, \Lambda, \bar{\eta}, R, \mathcal{J})$ of $S(K)(\mathbb{C})$ and (by reduction modulo p) a special point $(\tilde{A}, \tilde{\iota}, \tilde{\Lambda}, \bar{\tilde{\eta}}, R, \tilde{\mathcal{J}})$ of $S_{\mathfrak{p}}(K)(\bar{k})$. $\mathcal{A}$ is the isogeny class containing $(\tilde{A}, \tilde{\iota}, \tilde{\Lambda})$. We identify the contravariant rational Dieudonné module of A with $V \otimes k$

as above, then the F-translation is given by $x \mapsto \tilde{b}\sigma(x)$, where $\tilde{b} \in T(\mathfrak{k})$, futhermore $\tilde{b} = u^{-1}b\,\sigma(u)$, where b is constructed from ϕ (as in 1.2) and $u \in T(\mathfrak{k})$. In the first variant this follows from the fact that $\mathcal{D}$ is the gerb associated to the Tannakian category of isocrystals over $\mathfrak{k}$, and that the association of the contravariant rational Dieudonné module to a motive in $M_{\bar{\kappa}}$ corresponds to the operation of composing a representation of $\mathcal{P}$ with a homomorphism $\mathcal{D} \longrightarrow \mathcal{P}$ which is equvivalent to $\mathcal{J}_p : \mathcal{G} \rightarrow \mathcal{P}$ (LR, p. 162), and in the second variant this is the meaning of the mentioned theorem of Kottwitz.

If $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\tilde{\eta}}) \in \mathcal{A}$ and if α is an isogeny from $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ to $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$, then we can construct an element $(x_p,x^p) \in X_p \times X^p/K^p$ as follows: α is the composite of an isogeny α_p whose degree is divisible by p and an isogeny α^p whose degree is prime to p, α_p induces a homomorphism from the contravariant Dieudonné module of $\tilde{A}$ into $V \otimes \mathfrak{k}$, let M' be the image of this, then M' is a lattice of $V \otimes \mathfrak{k}$ and $M' = g(V_{\mathbb{Z}} \otimes \mathcal{O}_{\mathfrak{k}})$ for some $g \in G(\mathfrak{k})$. If we take $x_p = ugx_o \in G(\mathfrak{k})x_o$ (see 1.2), then $x_p \in X_p$. α^p is in fact an isomorphism between $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ and $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$, and since $\bar{\tilde{\eta}}^{-1}$ can be regarded as an element of X^p/K^p, $\bar{\tilde{\eta}}'$ determines an element x^p of X^p/K^p. The class of (x_p,x^p) in $I_\phi \backslash (X_p \times X^p/K^p)$ is independent of the choise of α, and the map $\mathcal{A} \longrightarrow I_\phi \backslash (X_p \times X^p/K^p)$ is a bijection (remark that we have an isomorphism $I_\phi \xrightarrow{\sim} \mathrm{Aut}(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ and that u determines an isomorphism $J'_\phi \xrightarrow{\sim} \mathrm{Aut}(V \otimes \mathfrak{k}, \iota, \{\psi\}))$.

The Frobenius action (over κ) on $\mathcal{A}$ is given by $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\bar{\tilde{\eta}}) \mapsto (\tilde{A}^{(q)},\tilde{\iota}^{(q)},\tilde{\Lambda}^{(q)},\bar{\tilde{\eta}}^{(q)})$ (the inverse image by the Frobenius over κ) and if we as isogeny from $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ to $(\tilde{A}^{(q)},\tilde{\iota}^{(q)},\tilde{\Lambda}^{(q)})$ choose α composed with the Frobenius isogeny from $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ to $(\tilde{A}^{(q)},\tilde{\iota}^{(q)},\tilde{\Lambda}^{(q)})$, then the lattice of $V \otimes \mathcal{k}$ associated to $\tilde{A}'^{(q)}$ is the image of M' by the **r-th** power of the F-translation, that is $(\tilde{b} * \sigma)^r M'$, and the element of X^p/K^p associated to $\bar{\tilde{\eta}}'^{(q)}$ is (by the definition of $\bar{\tilde{\eta}}'^{(q)}$) that associated to $\bar{\tilde{\eta}}'$. The Frobenius action on $\mathcal{A}$ is therefore given by the action of $\Phi = (b * \sigma)^r$ on X_p.

This bijection between the set of equvivalence classes of permissible homomorphisms $\phi : \mathcal{Q} \longrightarrow \mathcal{G}$ and the set of isogeny classes of $S_{\mathfrak{p}}(K)(\bar{k})$ can be refined to a bijection between the set of equvivalence classes of j-K-permissible pairs (ϕ, ε) and the set of j-isogeny classes of $S_{\mathfrak{p}}(K)(k^j)$. A j-permissible pair (ϕ, ε) is <u>j-K-permissible</u> if $(I_\phi)_\varepsilon \backslash (Y_p^j * Y^p)$ (see 1.2) is non-empty, that is, if 1) $\exists x \in X_p$: $\varepsilon' x = \Phi^j x$ and 2) $\exists y \in X^p$: $y^{-1} \varepsilon\, y \in K^p$ (see 1.3). Two j-K-permissible pairs (ϕ, ε) and (ϕ', ε') are equvivalent if $\phi' = \mathrm{ad}\, g \circ \phi$ and $\varepsilon' = \mathrm{ad}\, g(\varepsilon) \cdot z$ for some $g \in G(\bar{\mathbb{Q}})$ and $z \in Z(\mathbb{Q})_K$. If $(\tilde{A}, \tilde{\iota}, \tilde{\Lambda}, \bar{\tilde{\eta}})$ and $(\tilde{A}', \tilde{\iota}', \tilde{\Lambda}', \bar{\tilde{\eta}}')$ belongs to $S_{\mathfrak{p}}(K)(k^j)$ then an <u>j-isogeny</u> from $(\tilde{A}, \tilde{\iota}, \tilde{\Lambda})$ to $(\tilde{A}', \tilde{\iota}', \tilde{\Lambda}')$ is an isogeny which **commute with the Frobenius endomorphisms over** k^j **on** $\tilde{A}$ and $\tilde{A}'$. The j-isogeny class corresponding to (ϕ, ε) is that containing the point $(\tilde{A}, \tilde{\iota}, \tilde{\Lambda}, \bar{\tilde{\eta}})$ of $S_{\mathfrak{p}}(K)(k^j)$ constructed as follows: We can assume that $\phi = \psi_{T, \mu_h}$ and $\varepsilon \in T(\mathbb{Q})$ (LR, Lemma 5.23). Let $v \in T(\bar{\mathbb{Q}}_p)$ and $b \in T(\mathcal{k})$ be constructed from $\phi \circ \zeta_p$ as in 1.2. Choose $g_p \in G(\mathcal{k})$ such that for $x = g_p \cdot x_o$ is $\varepsilon' x = \Phi^j x$ and $y \in X^p$ such that $y^{-1} \varepsilon\, y \in K^p$, and if the F-translation on the contravariant rational Dieudonné module $V \otimes \mathcal{k}$ of $\tilde{A}$ (constructed from (T,h)) is given by $x \mapsto \tilde{b}\, \sigma(x)$ where $\tilde{b} \in T(\mathcal{k})$, choose $u \in T(\mathcal{k})$ such that $b = u \tilde{b} \sigma(u)^{-1}$. Let $g \in G(\mathbb{A}_f)$ be defined by $g_p = v^{-1} u^{-1} g_p$ and $g^p = y$.

To (T,h,g) we have constructed a special point $(A,\iota,\Lambda,\bar{\eta},R,\vartheta)$ of $S(K)(\mathbb{C})$ and (by reduction modulo p) a special point $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\tilde{\bar{\eta}},R,\tilde{\vartheta})$ of $S_{\wp}(K)(\bar{k})$, $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\tilde{\bar{\eta}})$ belongs to $S_{\wp}(K)(k^j)$ and the j-isogeny class of $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\tilde{\bar{\eta}})$ is independent of the choises. The lattice $L = g\cdot V_{\mathbb{Z}}$ of V (and the complex structure on $V\otimes\mathbb{R}$ given by h) defines an abelian variety A_0 over $\mathbb{C}$ in the isogeny class of (A,ι,Λ) (namely $A_0 = ((V\otimes\mathbb{R})/L)^*$), and since $\varepsilon \in G(\mathbb{Q})$ and $\varepsilon L \subset L$, ε defines an isogeny on (A_0,ι,Λ) and the reduction of this to $(\tilde{A},\tilde{\iota},\tilde{\Lambda})$ is the Frobenius endomorphism over k^j.

The above bijection between the isogeny class corresponding to ϕ and $I_{\phi}\backslash(X_p \times X^p/K^p)$ has in the present setting as analogous a bijection between the j-isogeny class $\mathcal{A}$ corresponding to (ϕ,ε) and $(I_{\phi})_{\varepsilon}\backslash(Y_p^j \times Y^p)$: if in the above proof we choose α such that it transforms the Frobenius endomorphism (over k^j) on $\tilde{A}$ to $\varepsilon \in I_{\phi}$, then x_p belongs to $Y_p^j \subset X_p$ and x^p belongs to $Y^p \subset X^p/K^p$, the class of (x_p,x^p) in $(I_{\phi})_{\varepsilon}\backslash(Y_p^j \times Y^p)$ is independent of the choise of α, and the map $\mathcal{A} \longrightarrow (I_{\phi})_{\varepsilon}\backslash(Y_p^j \times Y^p)$ is a bijection.

A <u>j-triple</u> $(\varepsilon,\delta,\gamma)$ consist of a $\varepsilon \in G(\mathbb{Q})_{s.s.}$ which is elliptic at infinety, a $\delta \in G(F^n)$ $(n = jr)$ such that $Nm_{F^n/\mathbb{Q}_p}\delta$ is stably conjugate to ε and a $\gamma_\ell \in G(\mathbb{Q}_\ell)$ (for each $\ell \neq p$) such that γ_ℓ is stably conjugate to ε (and conjugate to ε for almost all ℓ). The j-triples $(\varepsilon,\delta,\gamma)$ and $(\varepsilon',\delta',\gamma')$ are equvivalent if ε and ε' are stably conjugate, δ and δ' are $G(F^n)$-σ-conjugate, and γ and γ' are conjugate, and they are K-equvivalent if $(\varepsilon',\delta',\gamma')$ is equvivalent to $(\varepsilon z,\delta w,\gamma z)$, where $z \in Z(\mathbb{Q})_K$ and $w \in Z(F^n) \cap K_p(\mathcal{O}_{F^n})$ satisfies $Nm_{F^n/\mathbb{Q}_p} w = z$. We will not distinguish between a j-triple and its equvivalence class.

The Kottwitz invariant of a j-triple $(\varepsilon,\delta,\gamma)$ is the element $\beta(\delta,\gamma) \in \mathcal{K}(G_\varepsilon/\mathbb{Q})^D$ (see 1.7)(if $\varepsilon \sim_K \varepsilon'$ (stable conjugacy modulo $Z(\mathbb{Q})_K$) we can identify $\mathcal{K}(G_\varepsilon/\mathbb{Q})^D$ and $\mathcal{K}(G_{\varepsilon'}/\mathbb{Q})^D$, and K-equvivalent $(\varepsilon,\delta,\gamma)$ and $(\varepsilon',\delta',\gamma')$ have equal Kottwitz invariants (LR, Lemma 5,18)).

To an equvivalence class of j-permissible pairs $(\varphi,\bar{\varepsilon})$ we have (in 1.3) constructed an equvivalence class of j-triples $(\varepsilon,\delta,\gamma)$. The Kottwitz invariant of a such j-triple is 1, and conversely: any j-triple whose Kottwitz invariant is 1 is the j-triple of a j-permissible pair (LR, Satz 5,25), precisely $i(\varepsilon)$ inequvivalent j-permissible pairs have the same equvivalence class of j-triples $(\varepsilon,\delta,\gamma)$. Therefore we can to every j-isogeny class $\mathcal{A}$ of $S_\varphi(K)(k^j)$ associate a K-equvivalence class of j-triples $(\varepsilon,\delta,\gamma)$, namely that associated to the equvivalence class of j-K-permissible pairs corresponding to $\mathcal{A}$. The K-equvivalence class of j-triples of the j-isogeny class containing $(\tilde{A},\tilde{\iota},\tilde{\Lambda},\tilde{\bar{\eta}}) \in S_\varphi(K)(k^j)$ can be constructed directly as follows: The Frobenius endomorphism on $\tilde{A}$ (over k^j) determines an automorphism $\tilde{\varepsilon}$ of $V\otimes\bar{\mathbb{Q}}$, it belongs to $G(\bar{\mathbb{Q}})$ and can be chosen conjugate to an element $\varepsilon \in G(\mathbb{Q})_{s.s.}$. If the F-translation on the contravariant rational Dieudonné module $V\otimes k$ of $\tilde{A}$ is given by $x \mapsto \tilde{b}\sigma(x)$ $(\tilde{b} \in G(k))$, then $\tilde{b} = \tilde{\varepsilon}\,\tilde{b}\,\sigma(\tilde{\varepsilon})^{-1}$ (remark that $\tilde{\varepsilon} \in G(\mathbb{Q}_p^{un})$ because it is conjugate to ε) and we must have $Nm_{F^n/\mathbb{Q}_p}\,\tilde{b} = \hat{\varepsilon}\, c^{-1}\sigma^n(c)$ for some $c \in G(k)$, we take $\delta = c\tilde{b}\sigma(c)^{-1}$ (then $\delta \in G(F^n)$). Finally the Frobenius endomorphism (over k^j) on $\tilde{A}$ determines via a $\eta \in \tilde{\bar{\eta}}$ an automorphism γ_ℓ of $V\otimes\mathbb{Q}_\ell$ (for $\ell \neq p$), this belongs to $G(\mathbb{Q}_\ell)$(in fact it is conjugate to an element of K_ℓ).

A long step toward a proof of the conjecture in the general case would were taken if we to every point of $S_{\mathfrak{g}}(K)(k^j)$ can construct a K-equvivalence class of j-triples and prove that its Kottwitz invariant is 1.

3.2 Let G be an unramified connected reductive $\mathbb{Q}_p$-group (such that G_{der} is simply connected), let K be a hyperspecial subgroup and let F be an unramified extension of $\mathbb{Q}_p$ of degree n.

Let $\bar{M}$ be a G(F)-conjugacy class of homomorphisms $\mathbb{G}_m \longrightarrow G_F$ such that one (and so all) of the representations of $\mathbb{G}_m$ on $Lie(G_{\bar{\mathbb{Q}}_p})$ constructed from homomorphisms in $\bar{M}$ has no other weights than $o,\pm 1$, let $\tilde{f} \in \mathcal{H}(G(F),K(\mathcal{O}_F))$ be the characteristic function of the coset in $K(\mathcal{O}_F)\backslash G(F)/K(\mathcal{O}_F)$ corresponding to $\bar{M}$ (see 1.2) and let $f \in \mathcal{H}(G(\mathbb{Q}_p),K(\mathbb{Z}_p))$ be the image of $\tilde{f}$ by the base-change homomorphism (characterized by the property that $tr\,\pi_{\varphi}(f) = tr\,\pi_{\varphi'}(\tilde{f})$, where $\varphi' = \varphi|_{Gal(\mathbb{Q}_p^{un}/F)}$, for every admissible homomorphism $\varphi : Gal(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \longrightarrow {}^LG^o \rtimes Gal(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$).

If $\mathcal{E} \in G(\mathbb{Q}_p)^n$ (defined as in 1.4 but w.r.t. $\bar{M}$), let T be an elliptic Cartan subgroup of $G_{\mathcal{E}}$ and let $\mu \in X_*(T)$ be $M_{\mathcal{E}}$-conjugate to a μ satisfying the condition in 1.4, then the element $b_{\mathcal{E}} \in T(\mathcal{k})$ constructed from the homomorphism $\xi_{\mu}: \mathcal{D} \longrightarrow T(\bar{\mathbb{Q}}_p) \rtimes Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$(see 1.7) satisfies $Nm_{F/\mathbb{Q}_p}\, b_{\mathcal{E}} = \mathcal{E}\, c^{-1}\sigma^n(c)(c \in G(\mathcal{k}))$, and if $\delta_{\mathcal{E}} = cb_{\mathcal{E}}\sigma(c)^{-1}$ then $\delta_{\mathcal{E}} \in G(F)$(and $Nm_{F/\mathbb{Q}_p}\,\delta_{\mathcal{E}} = c\,\mathcal{E}c^{-1}$) and we have

$$c(G_{\mathcal{E}})\ O(\mathcal{E},f) = c(G^{\sigma}_{\delta_{\mathcal{E}}})\ TO(\delta_{\mathcal{E}},\tilde{f}) \qquad (*)$$

($G^{\sigma}_{\delta_\varepsilon}$ is an inner form of G_ε, this allow us to choose compatible measures on $G^{\sigma}_{\delta_\varepsilon}(\mathbb{Q}_p)$ and $G_\varepsilon(\mathbb{Q}_p)$).

If $\varepsilon \in G(\mathbb{Q}_p)_{s.s.} \setminus G(\mathbb{Q}_p)^n$ then $O(\varepsilon, f) = o$.

((✱) is proved in K7 for $\overline{M}$ trivial (that is, $\tilde{f}$ and f the unit elements) and in AC for $G = GL(n)$ and arbitrary $\tilde{f} \in \mathcal{H}(G(F), K(\mathcal{O}_F))$, $\varepsilon \in G(\mathbb{Q}_p)_{s.s.}$ and $\delta \in G(F)$ such that ε is conjugate (in $G(F)$) to $Nm_{F/\mathbb{Q}_p}\delta$ - in fact, this result is conjectured true for general G if orbital- resp. twisted orbital integral is replaced by stable orbital- resp. stable twisted orbital integral - in this case $SO(\varepsilon, f) = o$ if $\varepsilon \in G(\mathbb{Q}_p)_{s.s.}$ and not conjugate to a $Nm_{F/\mathbb{Q}_p}\delta$).

3.3 Let G be as in this paper and let $(H, s, \eta) \in \mathcal{E}$. For $\gamma \in H(\mathbb{Q})_{e,(G,H)\text{-reg}}$ and $\varepsilon \in G(\mathbb{Q})_e$ such that γ is the image of ε, we have

$$i(\gamma)\ |\mathcal{K}(H_\gamma/\mathbb{Q})|^{-1}\tau(H_\gamma)\ \tau(H)^{-1} = i(\varepsilon)\ |\mathcal{K}(G_\varepsilon/\mathbb{Q})|^{-1}\tau(G_\varepsilon)\ \tau(G)^{-1},$$

$\tau(H_\gamma)$ and $\tau(G_\varepsilon)$ are as defined in 1.6, and the measures on $H_\gamma(\mathbb{A})$ and $G_\varepsilon(\mathbb{A})$ are chosen compatible (recall that H_γ is an inner form of G_ε) - this measure on $H_\gamma(\mathbb{A})$ (and an arbitrary measure on $H(\mathbb{A})$) is used to define orbital integral on H. $\tau(H)$ and $\tau(G)$ are the Tamagawa numbers (proved in K6 for regular elements - we have used that Kottwitz in K8 has proved that $\tau(G) = 1$ for G simply connected semi-simple (if G has no E_8 factor)).

3.4 Let G be a connected reductive $\mathbb{R}$-group (such that G_{der} is simply connected) which has discrete series representations, let T be a fundamental Cartan subgroup and let ξ be a rational representation of G. For each $\varepsilon \in G(\mathbb{R})_e$ we choose a measure on $G_\varepsilon(\mathbb{R})$ such that the measures on $G_\varepsilon(\mathbb{R})$ and $G_{\varepsilon'}(\mathbb{R})$ are compatible if ε and ε' are stably conjugate – then we have a measure on the compact (modulo $Z(\mathbb{R})$) inner form $G_\varepsilon^!(\mathbb{R})$ of $G_\varepsilon(\mathbb{R})$. We define $\alpha: G(\mathbb{R}) \to \mathbb{R}$ by

$$\alpha(\varepsilon) = \begin{cases} c(G_\varepsilon^!) \ \mathrm{tr}\,\xi(\varepsilon)/\mathrm{meas}(Z(\mathbb{R})\backslash G_\varepsilon^!(\mathbb{R})) & \text{if } \varepsilon \in G(\mathbb{R})_e \\ 0 & \text{if } \varepsilon \in G(\mathbb{R}) \backslash G(\mathbb{R})_e . \end{cases}$$

If ε' is stably conjugate to ε then $\alpha(\varepsilon') = \alpha(\varepsilon)$.

Let (H,s,η) be an endoscopic datum for G (we assume that $\eta(s) \in {}^L T^o$) for which there is an isomorphism $X_*(T) \leftrightarrow X^*({}^L T^o)$ such that this, the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on T and $\eta(s)$ determine (H,s,η), also choose an extension $\eta': {}^L H^o \rtimes W_{\mathbb{R}} \to {}^L G^o \rtimes W_{\mathbb{R}}$ of η and a transfer factor $\Delta(\cdot,\cdot)$.

There exist a function f_ξ^H on $H(\mathbb{R})$ such that

$$SO(\gamma, f_\xi^H)^{*} = \begin{cases} \Delta(\gamma,\varepsilon)\,\alpha(\varepsilon) & \text{if } \gamma \in H(\mathbb{R})_e \\ 0 & \text{if } \gamma \in H(\mathbb{R})_{s.s.} \backslash H(\mathbb{R})_e , \end{cases}$$

here $\varepsilon \in T(\mathbb{R})$ is chosen so that γ is the image of ε via the isomorphism $X_*(T) \leftrightarrow X^*({}^L T^o)$ (obvius for H an elliptic Cartan subgroup of G, proved in L7, §6 and Ca for $H = GL(2)$ and G an inner form of H).

If (H,s,η) is not elliptic we take $f_\xi^H \equiv 0$.

*) The measure on $H_\gamma(\mathbb{R})$ is of course that compatible with the measure on $G_\varepsilon(\mathbb{R})$ (H_γ is an inner form of G_ε).

3.5 Let G be as in 3.2, let (H,s,η) be an endoscopic datum for G and let $\phi \in \mathcal{H}(G(\mathbb{Q}_p),K)$ be the characteristic function of K.

If there exist $\gamma \in H(\mathbb{Q}_p)_{s.s.,(G,H)\text{-reg}}$ such that the sum below is non-zero, then H is unramified (proved in LL for H elliptic Cartan subgroup of $G = GL(2)$). We choose an extension η': ${}^LH^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \longrightarrow {}^LG^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ of η, and we can choose a hyperspecial subgroup K^H of $H(\mathbb{Q}_p)$ such that every $\gamma \in K^H$ is the image of a $\varepsilon \in K$.

There exist a function $\phi^H \in \mathcal{H}(H(\mathbb{Q}_p),K^H)$ such that if $\gamma \in H(\mathbb{Q}_p)_{s.s.,(G,H)\text{-reg}}$ then

$$SO(\gamma,\phi^H) = \begin{cases} \Delta(\gamma,\varepsilon) \sum_{\mathcal{G} \in \mathcal{E}(G_\varepsilon/\mathbb{Q}_p)} \kappa(\mathcal{G})\, c(G_{\mathcal{G}_\varepsilon})\, O(\mathcal{G}_\varepsilon,\phi) & \text{if } \gamma \text{ is the image of } \varepsilon \in G(\mathbb{Q}_p)_{s.s.} \\ 0 & \text{if } \gamma \text{ is not the image of any } \varepsilon \end{cases}$$

(see 3.7).

Now we assume that $\eta(s)^m \in Z$ for some m.

Notation:

$\overline{M}$ is a $G(F)$-conjugacy class of homomorphisms $\mathbb{G}_m \longrightarrow G_F$ such that one (and so all) of the representations of $\mathbb{G}_m$ on $\mathrm{Lie}(G_{\overline{\mathbb{Q}}_p})$ constructed from homomorphisms in $\overline{M}$ has no other weights than $0, \pm 1$.

$\mathcal{R}_\mu \subset X^*({}^LT^o)$ is the Weyl-group orbit determined by $\overline{M}$.

${}^or$ is the (finite dimensional) representation of ${}^LG^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/F)$ (unique up to equvivalence) which is irreducible on ${}^LG^o$ having extreme ${}^LT^o$-weights $\mathcal{R}_\mu$ and for which $\mathrm{Gal}(\mathbb{Q}_p^{un}/F)$ acts trivially on the ${}^LB^o$-highest weight space.

r is ${}^or$ induced to ${}^LG^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$.

$n \in [F:\mathbb{Q}_p]\cdot \mathbb{N}$.

$f \in \mathcal{H}(G(\mathbb{Q}_p),K)$ is associated to the class function $x \mapsto \operatorname{tr} r(x^n)$ on ${}^LG^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ by the Satake transform.

$\gamma \in H(\mathbb{Q}_p)_{s.s.,(G,H)\text{-reg}}$ is the image of $\mathcal{E} \in G(\mathbb{Q}_p)^n$ (defined as in 1.4 but w.r.t. $\bar{M}$).

A Cartan subgroup T of $G_{\mathcal{E}}$ and an isomorphism $X_*(T) \longleftrightarrow X^*({}^LT^o)$ are chosen such that they arise from the correspondance between γ and $\mathcal{E}$ (see 1.9).

$\mu_o \in X_*(T)$ is $M_{\mathcal{E}}$-conjugate to a μ satisfying the condition in 1.4.

${}^or^H$ is the restriction of ${}^or$ to ${}^LH^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/F)$ (via η_p).

$\mathcal{X} = \{(\mu-\mu_o)(\eta(s)) \mid \mu \in \mathcal{Q}_\mu\} \subset$ roots of unity.

For $i \in \mathcal{X}$, ${}^or^{H,i}$ is the subrepresentation of ${}^or^H$ determined by $\{\mu \in \mathcal{Q}_\mu \mid (\mu-\mu_o)(\eta(s)) = i\}$.

$r^{H,i}$ is ${}^or^{H,i}$ induced to ${}^LH^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$.

$f_\gamma^H \in \mathcal{H}(H(\mathbb{Q}_p),K^H)$ is associated to the class function $x \mapsto \sum_{i \in \mathcal{X}} i \operatorname{tr} r^{H,i}(x^n)$ on ${}^LH^o \rtimes \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ by the Satake transform.

Then: f_γ^H is independent of the choise of $\mathcal{E}$ and we have

$$SO(\gamma, f_\gamma^H * \phi^H) = \Delta(\gamma,\mathcal{E}) \sum_{g \in \mathcal{E}(G_{\mathcal{E}}/\mathbb{Q}_p)} \kappa(g)\, c(G_{g_{\mathcal{E}}})\, O({}^g\mathcal{E}, f * \phi).$$

If $\mathcal{E} \in G(\mathbb{Q}_p)_{s.s.} \setminus G(\mathbb{Q}_p)^n$ then $O(\mathcal{E},f) = o$.

If $\gamma \in H(\mathbb{Q}_p)_{s.s.,(G,H)\text{-reg}}$ is not the image of any $\mathcal{E} \in G(\mathbb{Q}_p)^n$ then $SO(\gamma, f^H * \phi^H) = o$, here f^H is constructed as above but $\mu_o \in X^*({}^LT^o)$ is chosen arbitrary.

<u>3.6</u> Let G be as in 3.4. There exist a function f^G on $G(\mathbb{R})$ such that

$$SO(\varepsilon, f^G)^{*)} = \alpha(\varepsilon)$$

for $\varepsilon \in G(\mathbb{R})_{s.s.}$ (proved in L7, S6 and Ca for G = GL(2)).

Let (H, s, η) be as in 3.4, and let $\psi \in \Phi(H)$ be such that $\varphi = \eta' \circ \psi \in \Phi(G)_e$. We can assume that $\varphi(\mathbb{C}^*) \subset {}^LT^o \rtimes \mathbb{C}^*$ and $\varphi(\tau) = g \rtimes \tau$ where $g \in \mathrm{Norm}_{{}^LG^o}({}^LT^o)$. The action ι' on ${}^LT^o$ given by $\varphi(\tau)$ corresponds (via $X^*({}^LT^o) \leftrightarrow X_*(T)$) to the action on T given by the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, therefore ${}^LT^o \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for this action is the L-group of T. To φ is (by the Langlands correspondance, see Bo) associated a continuous regular character λ_o of $T(\mathbb{R})$ and so a discrete series representation π_o of $G(\mathbb{R})$, this belongs to $\Pi(\varphi)$ and we have

$$\sum_{\pi \in \Pi(\psi)} \langle 1, \pi \rangle \, \mathrm{tr}\, \pi(f^H) = e_\infty \langle \eta(s), \pi_o \rangle \sum_{\pi \in \Pi(\varphi)} \langle 1, \pi \rangle \, \mathrm{tr}\, \pi(f^G)$$

(for e_∞ and $\langle \cdot, \cdot \rangle$ see 3.7).

We can as a matter of course replace the isomorphism $X_*(T) \leftrightarrow X^*({}^LT^o)$ by the composite with a $\omega \in \Omega({}^LG^o, {}^LT^o) = \Omega(G(\mathbb{C}), T(\mathbb{C}))$ (because the action of ω on T is defined over $\mathbb{R}$), if we do so we must multiply f^H and $\langle \eta(s), \pi_o \rangle$ by $\kappa(\{\omega\}) (= \pm 1)$, where κ is the character of $H^1(\mathbb{R}, T) = \pi_o({}^LT^{o\,\Gamma_\infty})^D$ determined by $\{\eta(s)\} \in \pi_o({}^LT^{o\,\Gamma_\infty})$, as we note that $\Omega(G(\mathbb{C}), T(\mathbb{C}))/\Omega(G(\mathbb{R}), T(\mathbb{R})) = \mathfrak{D}(T/\mathbb{R}) \subset H^1(\mathbb{R}, T)$ and $\langle \eta(s), \pi_o^\omega \rangle = \kappa(\{\omega\}) \langle \eta(s), \pi_o \rangle$, here π_o^ω is attached to $\lambda_o \circ \omega$.

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*) The measure on $G_\varepsilon(\mathbb{R})$ is of course that entering the definition of α.

3.7 Let G be a connected reductive $\mathbb{Q}_v$-group (v place)(such that G_{der} is simply connected) and let (H,s,η) be an endoscopic datum for G. Choose an extension η': ${}^L H^o \rtimes L_{\mathbb{Q}_v} \longrightarrow {}^L G^o \rtimes L_{\mathbb{Q}_v}$ of η, and choose a transfer factor $\Delta_v(\gamma,\varepsilon)$.

There exist a $e_v \in \mathbb{C}^\times$ such that the following is true: if the functions f on $G(\mathbb{Q}_v)$ and f^H on $H(\mathbb{Q}_v)$ are connected by

$$SO(\gamma,f^H) = \begin{cases} \Delta_v(\gamma,\varepsilon) \sum_{\mathcal{S} \in \mathfrak{E}(G_\varepsilon/\mathbb{Q}_v)} \kappa(\mathcal{S})\, c(G_{\mathcal{S}_\varepsilon})\, O(\mathcal{S}_\varepsilon, f) & \text{if } \gamma \text{ is the image of } \varepsilon \in G(\mathbb{Q}_v)_{s.s.} \\ 0 & \text{if } \gamma \text{ is not the image of any } \varepsilon \end{cases}$$

(here $\gamma \in H(\mathbb{Q}_v)_{s.s.,(G,H)-reg}$), then we have for each $\psi \in \Phi(H)_{temp}$ such that $\varphi = \eta' \circ \psi \in \Phi(G)$:

$$\sum_{\pi \in \Pi(\psi)} \langle 1,\pi \rangle \operatorname{tr} \pi(f^H) = e_v \sum_{\pi \in \Pi(\varphi)} \langle \eta(s),\pi \rangle \operatorname{tr} \pi(f),$$

$\langle \cdot,\cdot \rangle$ is the usual pairing $\mathcal{S}_\varphi \times \Pi(\varphi) \longrightarrow \mathbb{C}$, where $\mathcal{S}_\varphi = S_\varphi/(S_\varphi)^o Z = \pi_0(S_\varphi/Z)$ and $S_\varphi = \{g \in {}^L G^o \mid \operatorname{ad} g \circ \varphi = \varphi\}$, $\langle \cdot,\cdot \rangle$ is not canonical, but this does not matter, since the global $\langle \cdot,\cdot \rangle$ which is the product of all the local $\langle \cdot,\cdot \rangle$ is canonical.

For a given function f on $G(\mathbb{Q}_v)$(smooth and of compact support) we can construct a function f^H on $H(\mathbb{Q}_v)$ such that f and f^H are connected as above (see LL for G = GL(2), LS2 for G a form of SL(3) and Sh for $v = \infty$).

3.8 Let G, f^G be as in 3.6, and let $\varphi \in \Phi(G)_{temp}$. If $\sum_{\pi \in \Pi(\varphi)} \mathrm{tr}\, \pi(f^G) \neq 0$, then φ is elliptic and $\Pi(\varphi)$ is the L-packet of discrete series representations of $G(\mathbb{R})$ associated to one of the absolutly irreducible components $\check{\xi}_{\Pi(\varphi)}$ of $\check{\xi}$, futhermore we have $\sum_{\pi \in \Pi(\varphi)} \mathrm{tr}\, \pi(f^G) = (-1)^d * \text{multiplicity of } \check{\xi}_{\Pi(\varphi)}$ in $\check{\xi}$ (this result is used only in the conclusion).

Let G, H, f^H be as in 3.4, and let $\psi \in \Phi(H)_{temp}$. If $\sum_{\pi \in \Pi(\psi)} \mathrm{tr}\, \pi(f^H) \neq 0$, then ψ and $\varphi = \eta' \circ \psi$ are elliptic (and so φ is admissible for G).

Let G,H be as in 3.7, and let the function ϕ^H on $H(\mathbb{Q}_v)$ be connected with the characteristic function ϕ of K (compact open subgroup of $G(\mathbb{Q}_v)$). If $\sum_{\pi \in \Pi(\psi)} \langle 1, \pi \rangle \, \mathrm{tr}\, \pi(\phi^H) \neq 0$, then $\varphi = \eta' \circ \psi$ is admissible for G.

3.9 Let G be as in this paper and let $(H, s, \eta) \in \mathcal{E}$. Let $\bar{\gamma} \in H(\mathbb{Q})_{e,(G,H)\text{-reg}}$ and $\bar{\varepsilon} \in G(\mathbb{Q})_e$ be such that $\bar{\gamma}$ is the image of $\bar{\varepsilon}$. Choose the local transfer factors $\Delta_v(\cdot,\cdot)$ such that $\Delta_v(\bar{\gamma},\bar{\varepsilon}) = 1$ for almost all places v and $\prod_v \Delta_v(\bar{\gamma},\bar{\varepsilon}) = 1$. Then $e_v = 1$ for almost all places v and $\prod_v e_v = 1$.

3.10 We assume that (a sufficiently large part of) the Langlands correspondance has been constructed - that is, for a given reductive algebraic group, we have a map (having the expected properties) from the equvivalence classes of admissible homomorphisms from the Weil- (or rather the Langlands-) group into the L-group associated to the group to the L-packets of representations of the group - the map is a bijection in the local case and maps to automorphic representations in the global case.

Let G be a connected reductive $\mathbb{Q}$-group, let ${}_oZ$ be a closed subgroup of $Z(\mathbb{A})$ of the form $\prod_v {}_oZ_v$ (Z center of G) such that ${}_oZZ(\mathbb{Q})$ is closed in $Z(\mathbb{A})$ and ${}_oZZ(\mathbb{Q})\backslash Z(\mathbb{A})$ is compact, let χ be a character of $({}_oZ\cap Z(\mathbb{Q}))\backslash {}_oZ$ and let $\Phi(G)_e$ be the set of (equvivalence classes of) elliptic tempered admissible homomorphisms $\varphi: L_{\mathbb{Q}} \longrightarrow {}^LG^o \rtimes L_{\mathbb{Q}}$ such that $\chi_\varphi|_{{}_oZ} = \chi$ ($L_{\mathbb{Q}}$ is the Langlands group, it is an extension of $W_{\mathbb{Q}}$ by a compact group, see L5 and K3). Then the stable tempered cuspidal part of the trace is

$$d_\varphi^{-1} \sum_{\varphi\in\Phi(G)_e} \sum_{\pi\in\Pi(\varphi)} n_\pi \operatorname{tr}\pi(f)$$

(see K3), d_φ is the number of (global) equvivalence classes in the local equvivalence class of φ (d_φ different classes of $\Phi(G)_e$ parametrize $\Pi(\varphi)$) and $n_\pi = d_\varphi|\mathcal{S}_\varphi|^{-1}\langle 1,\pi\rangle$ is the "stable multiplicity" of π (for all this see LL)(f is assumed to be of the form $f = \prod_v f_v$ and to satisfy $f(zg) = \chi(z)^{-1}f(g)$ for $z\in {}_oZ$, and $\pi(f) = \int_{Z\backslash G(\mathbb{A})} \pi(g)\, f(g)\, dg$).

This part of the stable trace is "contained" in the stable elliptic part of the trace.