2. The formal part of the proof.

$$\sum_{\mathfrak{p}|p} \log Z(s, S_{\mathfrak{p}}(K), \xi)$$

$$= \sum_{\mathfrak{p}|p} \sum_{j=1}^{\infty} \frac{|\omega_{\mathfrak{p}}|^{js}}{j} \sum_{x \in S_{\mathfrak{p}}(K)(\mu^{j})} \mathrm{tr}(\Phi_{\mathfrak{p}}^{j})_x \tag{1}$$

$$= \text{——} \;''\; \text{——} \sum_{\emptyset} \sum_{\varepsilon \in I_{\emptyset}/\sim_K} \mathrm{tr}\,\xi(\varepsilon) \left| (I_{\emptyset})_{\varepsilon} \backslash (Y_p^{j} \times Y^{p}) \right| \tag{2}$$

$$= \text{——} \;''\; \text{——} \sum_{\substack{\{\emptyset,\varepsilon\} \\ j\text{-perm.} \\ K\text{-equ. cl.}}} \mathrm{tr}\,\xi(\varepsilon)\; \mathrm{meas}((I_{\emptyset})_{\varepsilon} Z_K \backslash (G_{\delta}^{\sigma}(\mathbb{Q}_p) \times G_{\gamma}(\mathbb{A}_f^{p}))) \times TO(\delta, \tilde{f}_{\mathfrak{p},n})\; O(\gamma, \phi^{p}) \tag{3}$$

$$= \text{——} \;''\; \text{——} \sum_{\varepsilon \in G(\mathbb{Q})^{n}_{\infty}/\sim_K} \mathrm{tr}\,\xi(\varepsilon) \sum_{\substack{\{\emptyset,\varepsilon'\} \\ j\text{-perm.} \\ K\text{-equ. cl.}}}{}_{\varepsilon' \sim_K \varepsilon} \text{——} \;''\; \text{——} \tag{4}$$

$$= \text{——} \;''\; \text{——} \sum_{\varepsilon \in \left\{ \begin{smallmatrix} \text{fav.} \\ \text{rep.} \end{smallmatrix} \right\}} c_{\infty}\, \mathrm{tr}\,\xi(\varepsilon) \sum_{\emptyset \in \mathcal{P}_{\varepsilon}} \mathrm{meas}((G_{\varepsilon})_{\emptyset}(\mathbb{Q}) Z_K \backslash (G_{\varepsilon})_{\emptyset}(\mathbb{A}_f)) \times c_p\; TO(\delta, \tilde{f}_{\mathfrak{p},n})\; c^{p}\; O(\gamma, \phi^{p}) \tag{5}$$

$$= \text{——} \;''\; \text{——} \sum_{\varepsilon \in \left\{ \begin{smallmatrix} \text{fav} \\ \text{rep} \end{smallmatrix} \right\}} \alpha(\varepsilon)\, \tau(G_{\varepsilon})_K \sum_{\emptyset \in \mathcal{P}_{\varepsilon}} c_p\; TO(\delta, \tilde{f}_{\mathfrak{p},n})\; c^{p}\; O(\gamma, \phi^{p}) \tag{6}$$

$$= \sum_{\mathfrak{p}|p} \sum_{j=1}^{\infty} \frac{|\varpi_{\mathfrak{p}}|^{js}}{j} \sum_{\varepsilon \in \left\{\substack{\text{fav.}\\ \text{rep.}}\right\}} \alpha(\varepsilon)\tau(G_{\varepsilon})_K \, i(\varepsilon) \, |\mathcal{K}(G_{\varepsilon}/\mathbb{Q})|^{-1} \sum_{\kappa \in \mathcal{K}(G_{\varepsilon}/\mathbb{Q})} \kappa_{\infty}(\mu - \mu_h)$$
$$\times \sum_{\delta \in \mathcal{E}(G_{\varepsilon}/\mathbb{Q}_p)} \kappa_p(\delta) \, c(G_{\delta_{\varepsilon}}) \, O(\delta_{\varepsilon}, f_{\mathfrak{p},n})$$
$$\times \sum_{\delta \in \mathcal{E}(G_{\varepsilon}/\mathbb{A}_f^p)} \kappa^p(\delta) \, c(G_{\delta_{\varepsilon}}) \, O(\delta_{\varepsilon}, \phi^p) \tag{7}$$

$$= \text{——} \,"\, \text{——} \, (1/r) \sum_{\varepsilon \in G(\mathbb{Q})^n_{\infty}/N_K} \; \sum_{\kappa \in \mathcal{K}(G_{\varepsilon}/\mathbb{Q})} i(\varepsilon) |\mathcal{K}(G_{\varepsilon}/\mathbb{Q})|^{-1} \tau(G_{\varepsilon})_K$$
$$\times (\Delta_{\infty}(\gamma,\varepsilon)\,\alpha(\varepsilon)\,\kappa_{\infty}(\mu - \mu_h))$$
$$\times (\Delta_p(\gamma,\varepsilon) \; r \sum \cdots)$$
$$\times (\Delta^p(\gamma,\varepsilon) \sum \cdots) \tag{8}$$

$$= \sum_{\mathfrak{p}|p} \sum_{\substack{j=1\\ r|j}}^{\infty} \frac{|\omega|^{js}}{j} \sum_{(H,s,\eta) \in \mathcal{E}_{\infty}} \iota(G,H) \sum_{\gamma \in H(\mathbb{Q})^j_{\infty}/N_K} i(\gamma) \, |\mathcal{K}(H_{\gamma}/\mathbb{Q})|^{-1} \tau(H_{\gamma})_K$$
$$\times SO_{\infty}(\gamma, f^H_{\infty}) \; SO_p(\gamma, r \, f^H_{\mathfrak{p},j} * \phi^H_p) \; SO^p(\gamma, \phi^{Hp}) \tag{9}$$

$$= \text{——} \,"\, \text{——} \sum_{(H,s,\eta) \in \mathcal{E}} \iota(G,H) \sum_{\psi \in \Phi(H)_e} |\mathcal{S}_{\psi}|^{-1} \sum_{\pi \in \Pi(\psi)} \langle 1, \pi \rangle \, \mathrm{tr}\, \pi(F^H_{\mathfrak{p},j}) \tag{10}$$

+ terms from the non-tempered-cuspidal part
of the stable elliptic part of the trace

$$= \text{non-temp.-cusp. part of the zeta function} + \sum_{(H,s,\eta)\in\mathcal{E}} \iota(G,H) \sum_{\psi\in\Phi(H)_{G-e}} |\mathcal{S}_\psi|^{-1} \Big(\sum_{\pi\in\Pi_\infty^H} \langle 1,\pi\rangle \operatorname{tr}\pi(f_\xi^H)\Big)$$

$$\times \Big(\sum_{\mathfrak{p}|p} \sum_{\substack{j=1 \\ r|j}}^{\infty} \frac{|\omega|^{js}}{j} \operatorname{tr}\pi_p(r\, f^H_{\mathfrak{p},j})\Big)$$

$$\times \Big(\sum_{\pi\in\Pi_f^H} \langle 1,\pi\rangle \operatorname{tr}\pi(\phi^H)\Big) \tag{11}$$

$$= \text{———}\;\text{"}\;\text{———} + \sum_{(H,s,\eta)\in\mathcal{E}} \iota(G,H) \sum_{\psi\in\Phi(H)_{G-e}} |\mathcal{S}_\psi|^{-1} m(\Pi_\infty)$$

$$\times \Big(\sum_{i\in\mathcal{H}} \langle \eta(s), \Pi_\infty^{i\eta}\rangle \log L(s-d/2, \pi_p, r^{H,i})\Big) \tag{12}$$

$$\times \Big(\sum_{\pi\in\Pi_f} \langle \eta(s),\pi\rangle \operatorname{tr}\pi(\phi)\Big)$$

$$= \text{———}\;\text{"}\;\text{———} + \sum_{\varphi\in\Phi(G)_e} \sum_{\{sZ\}\in\mathcal{S}_\varphi/\sim} \sum_{(H,s,\eta)\in\mathcal{E}} \lambda(H,s,\eta)^{-1} \sum_{\substack{\psi\in_G\Phi(H)_{G-e} \\ ((H,s,\eta),\psi)\mapsto(\varphi,\{sZ\})}}$$

$$|_G\mathcal{S}_\psi|^{-1} m(\Pi_\infty) \times \text{last two lines of (12)} \tag{13}$$

$$= \text{———}\;\text{"}\;\text{———} + \log \prod_{\varphi\in\Phi(G)_e} \prod_{sZ\in\mathcal{S}_\varphi} \Big(\prod_{i\in\mathcal{H}} L(s-d/2, \pi_p, r^{H,i})^{\langle s,\Pi_\infty^{i\eta}\rangle}\Big)^a$$

$$\text{where } a = |\mathcal{S}_\varphi|^{-1} m(\Pi_\infty) \Big(\sum_{\pi\in\Pi_f} \langle s,\pi\rangle \operatorname{tr}\pi(\phi)\Big) \tag{14}$$