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On the Zeta Function of a General Shimura Variety

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Print: BoD, 2022
Book-ID 0216944171

Introduction

LANGLANDS shows in his paper L6 how the zeta function of certain Shimura varieties can be expressed as a product of L-functions associated to automorphic representations of the algebraic group G entering the description of the Shimura variety (or rather, the endoscopic groups for G). The group G is here (roughly speaking) obtained by scalar reduction to $\mathbb{Q}$ of the multiplicative group of a certain quaternion algebra over an algebraic numberfield. The paper L6 is concerned with the local zeta function of the variety obtained by reducing the Shimura variety at a (finite) place of its definition field where it has good reduction, and it is based on a description of this reduced variety which was unproven (and which was formerly presented in L2 and L3 - a more detailled account can be found in M1 and M2).

L6 is a contribution to a theory which in some future should tell us how we can generalize some classical results, such as that (due to Eichler) saying that the zeta function of a modular curve $\Gamma\backslash H$ (H the upper halfplane and Γ some congruence subgroup of $SL_2(\mathbb{Z})$) can be expressed as a product of L-functions associated to automorphic forms on $\Gamma\backslash H$ (or otherwise speaking, to automorphic representations of $GL_2(\mathbb{A})$), can be analytically continued and that the analytic continuation satisfies a functional equation.

The proofs of the classical results are based on congruence relations between Hecke operators and the Frobenius, and this method does not seem to work for general Shimura varieties. The proof in L6 is based on the Selberg trace formula and is in some simpler cases presented in L3, Ca and La. (see also BL, HLR and Ra), but the cases studied in L6 take care of a complication that arises by the fact that whereas a L-function is associated not to a single representation but to an L-indistinguishable class of representations of $G(\mathbb{A})$, two L-indistinguishable representations can occur with different multiplicity in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$. This misforlune can be restored by using L-functions not associated to representations of G, but to representations of the so-called endoscopic groups for G. Even though the endoscopic groups in the cases studied in L6 are of a rather simple type, as they are either elliptic Cartan subgroups of G or the quasi-split inner form of G, L6 gives nevertheless a clear picture of the way in which they come into play in the general case.

Two circumstances however make it difficult immediately to generalize the method of L6. A class decomposition of the points of the reduced variety is parametrized by equivalence classes of so-called Frobenius pairs, but different domains can correspond to the same equivalence class because the equivalence relation is of local nature where it ought to be of global nature. Moreover the number of points left fixed by a power of the Frobenius is calculated explicitely by a complicated combinatoric argument.

In Langlands and Rapoport's paper LR the first difficulty is remedied - the description of the points conjectured there is more elegant and will possibly cover also the case of bad reduction (see Ra), and (expect for some standard conjectures of algebraic geometry) it is proved to be true in the case of good reduction for certain Shimura varieties that parametrize families of polarized abelian varieties with endomorphism- and level structure.

In Kottwitz's paper K4 - a special case is worked out of an idea which seems to make it possible to reduce all the combinatoric calculations in L6 to some standard problems in harmonic analysis: the relation between orbital- resp. twisted orbital integrals of associated functions in the case of passing to endoscopic groups resp. the case of base change.

In the present paper I will show - by using primary the material of LR and K4, and building on the ideas and techniques of L6 - how a proof for the expression of the "tempered cuspidal" part of the local zeta function in terms of L-functions in the case of a general Shimura variety should be set up: the proof will built on some precisely formulated conjectures of general nature. The purely formal part of the proof is presented in section 2, section 1 is devoted an explanation of each step of section 2, and section 3 is a list of all conjectures used.

It is necessarily to presuppose that the reductive $\mathbb{Q}$-group G is such that G_{der} is simply connected - why and how the general case can simply be reduced to this case is explained in LR. Moreover the Shimura variety in question is assumed to be of compact type, that is, its points with coordinates in $\mathbb{C}$ is a compact space, this amounts to demand that G_{ad} is anisotropic over $\mathbb{Q}$.

Contents

1 Explanation to each step in 2

1.1 Let G be a connected reductive algebraic group over $\mathbb{Q}$, and let X_∞ be a $G(\mathbb{R})$-conjugacy class of homomorphisms from $\underline{S} = res_{\mathbb{C}/\mathbb{R}}G_m$ into $G_\mathbb{R}$ such that if $h \in X_\infty$ then

1) the composition $G_m \to^w \underline{S} \to^h G_\mathbb{R}$ is central (w is the inclusion)

2) the Hodge structure on $Lie(G)(\mathbb{R})$ given by $\underline{S}(\mathbb{R}) = \mathbb{C}^\times \to^h G(\mathbb{R}) \to^{ad} Aut(Lie(G)(\mathbb{R}))$ is of type (-1, 1)+(0, 0)+(1, -1)

3) ad $h(i)$ (which is an involution on $G(\mathbb{R})$) induces a Cartan involution on $G_{der}(\mathbb{R})$

(if these conditions are satisfied by one $h \in X_\infty$, they are satisfied by all $h \in X_\infty$).

If $h \in X_\infty$ and if K_∞ denotes the centralizer of h in $G(\mathbb{R})$, then $K_\infty \cap G_{der}(\mathbb{R})^0$ is a maximal compact subgroup of $G_{der}(\mathbb{R})^0$ and X_∞ can be identified with $G(\mathbb{R})/K_\infty$.

If T is a Cartan subgroup of $G_\mathbb{R}$ and if $h \in X_\infty$ factorizes through T, then we have the composite μ_h: $G_m \to^{\iota 1} \underline{S}_\mathbb{C} \to^h T_\mathbb{C}$, thus $\mu_h \in X_*(T)$ ($\iota 1$ is given by $z \to (z, 1)$).

We can define a complex structure on X_∞ in the following way: for $h \in X_\infty$ we have a decomposition of the Lie algebra of $G(\mathbb{C})$

$$\mathfrak{g}_\mathbb{C} = \mathfrak{p}_h + \mathfrak{k}_h + \overline{\mathfrak{p}}_h$$

given by

$$ad(h(z_1, z_2))(X) = z_1^{-1}z_2\, X,\ X,\ \underline{z}_1 z_2^{-1}\, X$$
$$\text{for } X \in \text{resp. } \mathfrak{p}_h, \mathfrak{k}_h \text{ and } \overline{\mathfrak{p}}_h$$

($\mathfrak{k}_h$ is the complexifikation of the Lie algebra of the centralizer K'_∞ of h in $G(\mathbb{R})$ and $\mathfrak{p}_h$ resp. $\overline{\mathfrak{p}}_h$ is spanned by the root vectors attached to the positive resp. the negative non-compact roots of T for an order that puts μ_h into the negative closed Weyl chamber - h factorizes through T). Since $G(\mathbb{R})$ acts on the real manifold X_∞ (by conjugation), every vector $X \in g_{\mathbb{C}}$ defines a complex vector field $h \to X_h$ on X_∞, and the complex structure on X_∞ is such that the holomorphic, resp. antiholomorphic, space at h is $\bar{p}_h$ resp. p_h.

We choose an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and an imbedding $\bar{\mathbb{Q}} \to \mathbb{C}$, and we regard $\bar{\mathbb{Q}}$ as a subfield of $\mathbb{C}$.

Let T be a Cartan subgroup of G, let $h \in X_\infty$ factorizes through T, and let E denote the smallest Galois extension of $\mathbb{Q}$ (in $\bar{\mathbb{Q}}$) such that if $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$ then $\sigma\mu_h$ is within the $\Omega(G, T)$-orbit of μ_h. Then E is independent of the choice of T and h.

If we, for any field F containing $\mathbb{Q}$, let $\mathscr{M}(F)$ be the set of G(F)-conjugacy classes of homomorphisms $G_m \to G_F$, then X_∞ (via the assignment $h \to \mu_h$) gives rise to a class $\bar{M}_{\mathbb{C}}$ in $\mathscr{M}(\mathbb{C})$ (which is independent of the choice of T and h), this class in fact comes from a class $\bar{M}_{\bar{\mathbb{Q}}}$ in $\mathscr{M}(\bar{\mathbb{Q}})$ (K4) and E is the definition field of $\bar{M}_{\bar{\mathbb{Q}}}$, that is, the smallest Galois extension of $\mathbb{Q}$ such that Gal $(\bar{\mathbb{Q}}/E)$ leaves $\bar{M}_{\bar{\mathbb{Q}}}$ invariant.

We now assume that G_{ad} is anisotropic over $\mathbb{Q}$, and that K is a compact open subgroup of $G(\mathbb{A}_f)$. Then it is known (M3) that for K sufficiently small there exist one and only one (up to isomorphism over E) smooth and proper variety S(K) over E - the *Shimura variety* attached to the data G, X_∞, K - such that

1) $S(K)(\mathbb{C}) = G(\mathbb{Q})\backslash(X_\infty \times G(\mathbb{A}_f)/K)$ (this is a complex manifold since $G(\mathbb{A}_f)/K$ is discrete and $G(\mathbb{Q})$ acts freely on $X_\infty \times G(\mathbb{A}_f)/K$)

2) for any Cartan subgroup T of G and $h \in X_\infty$ such that

h factorizes through T, the following condition shall hold: let K_T denote $T(\mathbb{A}_f) \cap K$ and let E_h ($\subset \overline{\mathbb{Q}}$) denote the field of definition of μ_h ($\in X_*(T)$), then it is known that there exists one and only one (up to isomorphism over E_h) finite variety $S_h(K_T)$ over E_h such that

1) $S_h(K_T)(\mathbb{C}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T$

2) $Gal(E_h^{ab}/E_h)$ acts on $\pi_0(S_h(K_T)) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T$ through the inverse of the homomorphism $Gal(E_h^{ab}/E_h) = \pi_0(E_h^\times(\mathbb{Q})\backslash E_h^\times(\mathbb{A})) \to T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T$ defined by

$$E_h^\times \to^{Res\ \mu h} Res_{Eh/\mathbb{Q}} T_{Eh} \to^{NEh/\mathbb{Q}} T,$$

the imbedding $T \subset G$ defines a morphism $S_h(K_T)_\mathbb{C} \to S(K)_\mathbb{C}$,

the condition is now that this morphism shall be defined over $E \cdot E_h$ (D2).

Let ξ be a $\mathbb{Q}$-rational representation of G (acting on the $\mathbb{Q}$-vector space V), we can assume that ξ acts as a character ν on Z (the center of G).

For almost every prime ideal $\wp$ of E it will be true that S(K) has good reduction at $\wp$, that is, there is a smooth and proper scheme over $\mathcal{O}_{E\wp}$ whose base extension by $spec(E_p) \to spec(\mathcal{O}_{E\wp})$ is $S(K)_{E\wp}$. We assume that $\wp$ is a such prime ideal. Let p be the prime number in $\mathbb{Z} \cap \wp$. We thus have a smooth and proper variety $S_\wp(K)$, called the reduction of S(K) modulo $\wp$, over the finite field $\kappa = \mathcal{O}_{Ep}/\wp\mathcal{O}_{E\wp} = F_q$, for which the previous is the base-change by $\mathcal{O}_{E\wp} \to \kappa$, here $q = p^r$ and $r = [E_\wp:\mathbb{Q}_p]$ (independent of $\wp|p$ since E is Galois).

In order to define the zeta function of $S_\wp(K)$ w.r.t. the representation ξ we have need for a locally free sheaf of

$\mathbb{Q}_\ell$-vector spaces $F_{\xi,\wp}(K)$ over $S_\wp(K)(\bar\kappa)$ and an action of $Gal(\bar\kappa/\kappa)$ on $F_{\xi,\wp}(K)$ which commutes with the action of $Gal(\bar\kappa/\kappa)$ on $S_\wp(K)(\bar\kappa)$, here ℓ is an arbitrary prime number different from p.

This shaef is constructed in the following way (Ll): $G(\mathbb{Q}_\ell)$ acts on $V(\mathbb{Q}_\ell)$ by ξ. Let $V(\mathbb{Z}_\ell)$ be a compact open subgroup of $V(\mathbb{Q}_\ell)$ which is invariant under the action of K. If $V(\mathbb{Z}) = V(\mathbb{Q}) \cap V(\mathbb{Z}_\ell)$, then $V(\mathbb{Z})$ is a lattice in $V(\mathbb{Q})$ and $V(\mathbb{Z}_\ell) = V(\mathbb{Z}) \otimes \mathbb{Z}_\ell$. K acts on $V(\mathbb{Z}_\ell)/\ell^n V(\mathbb{Z}_\ell) = V(\mathbb{Z}/\ell^n\mathbb{Z})$ $(n \in \mathbb{N})$ (finite group). Let K_0 be a normal open subgroup of K acting trivially on $V(\mathbb{Z}/\ell^n\mathbb{Z})$. Then K/K_0 acts on $V(\mathbb{Z}/\ell^n\mathbb{Z})$. And K/K_0 acts also on $S(K_0)$ through morphisms defined over E (if $g \in G(\mathbb{A}_f)$ and $g^{-1}K'g \subset K$ then right multiplication by g will induce a map $S(K')(\mathbb{C}) \to S(K)(\mathbb{C})$ which is the map of points in $\mathbb{C}$ of a morphism $S(K') \to S(K)$ defined over E). The projection $S(K_0)(\mathbb{C}) \to S(K)(\mathbb{C})$ is the map of points in $\mathbb{C}$ of a morphism $S(K_0)(\mathbb{C}) \to S(K)$ defined over E. This morphism identifies S(K) with the quotient variety of $S(K_0)$ w.r.t. the action of K/K_0. $V(\mathbb{Z}/\ell^n\mathbb{Z}) \times_{K/K0} S(K_0)$ is a scheme over S(K). If we reduce this modulo $\wp$, then the set of points with coordinates in $\bar\kappa$ defines a locally free sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules over $S_\wp(K)(\bar\kappa)$ on which $Gal(\bar\kappa/\kappa)$ acts. If we take the limit for $n \to \infty$ and tensorize with $\mathbb{Q}_\ell$, we get the wanted sheaf $F_{\xi,\wp}(K)$ over $S_\wp(K)(\bar\kappa)$.

Let $\Phi_\wp$ denote the Frobenius in $Gal(\bar\kappa/\kappa)$ (and also a Frobenius element for $\wp$ in $Gal(\bar E/E)$). And let, for $j \in \mathbb{N}$, κ^j denote $F_{q^j} = F_{p^n}$, where n = jr. Then $S_\wp(K)(\kappa^j)$ is the set of fixed points for $\Phi_\wp^j$ on $S_\wp(K)(\bar\kappa)$, and for $x \in S_\wp(K)(\kappa^j)$ $\Phi_\wp^j$ will induce a linear endomorphism on the fibre of $F_{\xi,\wp}(K)$ over x, we denote this endomorphism by $(\Phi_\wp^j)_x$.

The zeta function of $S_{\wp}(K)$ *w.r.t.* ξ is now defined by

$$\log Z(s, S_{\wp}(K), \xi) = \Sigma_{j=1}^{\infty} |\omega_{\wp}|^{js}/j\ \Sigma\ tr(\Phi_{\wp}^{j})_x$$
$$\text{(sum over } x \in S_{\wp}(K)(\kappa^j))$$

($s \in \mathbb{C}$, Re $s >> 0$, $\omega_{\wp}$ is an uniformizer in $E_{\wp}$). If ξ is trivial

$$\Sigma_{j=1}^{\infty} |\omega_{\wp}|^{js}/j\ |S_{\wp}(K)(\kappa^j)| = \log \Pi\ (1 - |\omega_{\wp}|^{s\cdot \deg(x)})^{-1}$$
$$\text{(product over } x \in S_{\wp}(K))$$

(for Re $s >>0$), here $|S_{\wp}(K)|$ is the set of closed points (over κ) of $S_{\wp}(K)$ and $\deg(x) = [k(x):\kappa]$. If it was true that S(K) in reality was defined over $\mathcal{O}_E$, then $|S_{\wp}(K)|$ would be $|S(K)|_{\wp}$ (the set of closed points x (over $\mathcal{O}_E$) of S(K) for which the kernel of $\mathcal{O}_E \to k(x)$ is $\wp$) and we would have had

$$\Pi_{\wp \text{ prime of } E}\, Z(s, S_{\wp}(K)) = \Pi_{\ x \in |S(K)|}\ (1 - |k(x)|^{-s})^{-1}$$

which is the Hasse-Weil zeta function of S(K) (over $\mathcal{O}_E$) (strictly speaking the Hasse-Weil zeta function is the inverse of this).

Although we will not use cohomology for the calculation of $\Sigma\ tr(\Phi_{\wp}^{j})_x$ (sum over $x \in S_{\wp}(K)(\kappa^j)$), we will for later remarks have need for a formula which expresses this term in terms of the action of $\Phi_{\wp}$ on cohomology spaces.

We regard S(K) as being defined over $\overline{E}$. If p: U $\to$ S(K) is an étal covering of S(K), the set $\zeta_{\xi}(K)_{\mathbb{Z}/\ell^n\mathbb{Z}}(U, p)$ of sections of the base change by p of the scheme $V(\mathbb{Z}/\ell^n\mathbb{Z})\times_{K/K0}S(K_0)$ over S(K) has a $\mathbb{Z}/\ell^n\mathbb{Z}$-module structure and $\zeta_{\xi}(K)_{\mathbb{Z}/\ell^n\mathbb{Z}}$ is a locally free sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on the étal topology of $S(K)_{\overline{E}}$. By taking limit and tensoring with $\mathbb{Q}_\ell$ we get a locally free sheaf of $\mathbb{Q}_\ell$-vectorspaces on the étal topology of $S(K)_{\overline{E}}$.

Gal($\overline{E}$/E) acts on the $\mathbb{Z}/\ell^n\mathbb{Z}$-module $H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Z}/\ell^n\mathbb{Z}})$ ($0 \le i \le 2\dim S(K)$), and so on the $\mathbb{Q}_\ell$-vectorspace $\mathbb{Q}_\ell \otimes_{\mathbb{Z}\ell}$ $(\lim_{n\to\infty} H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Z}/\ell^n\mathbb{Z}})) = H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell})$. Because our assumptions on $\mathscr{p}$ the action of Gal($\overline{E}_{\mathscr{p}}$/$E_{\mathscr{p}}$) is unramified, the action of $\Phi_{\mathscr{p}}$ is well defined. By the Lefschetz fixed point formula we have

$$\Sigma\, tr(\Phi_{\mathscr{p}}^j) \text{ (sum over } x \in S_{\mathscr{p}}(K)(\kappa^j))$$
$$= \Sigma_{i=0}^{2\dim S(K)} (-1)^i\, tr\, \Phi_{\mathscr{p}}^j | H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell}).$$

We could consequently have defined the zeta function of $S_p(K)$ w.r.t. ξ by

$$Z(s, S_{\mathscr{p}}(K), \xi)$$
$$= \Pi_{i=0}^{2\dim S(K)} \det(1 - |\omega_{\mathscr{p}}|^s\, \Phi_{\mathscr{p}} | H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell}))^{(-1)\wedge(i+1)}$$

- the right hand side is a rational function in $|\omega_{\mathscr{p}}|^s$ with coefficients in $\mathbb{Z}$ (and independent of ℓ), therefore the right hand side has meaning (see D1).

If we choose a $h \in X_\infty$, then the set

$$G(\mathbb{Q})\backslash(\cup_{g\in G(\mathbb{A})} gV(\mathbb{Z})\times g))/K_\infty K,$$

where $gV(\mathbb{Z}) = V(\mathbb{Q}) \cap g_f V(\mathbb{Z}_f)$ ($g = g_\infty \cdot g_f$), defines a locally free sheaf of $\mathbb{Z}$-modules over $S(K)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K$ (and independent of the choise of h). If we tensorize this sheaf with $\mathbb{Z}/\ell^n\mathbb{Z}$, we get the sheaf over $S(K)(\mathbb{C})$ defined by $V(\mathbb{Z}/\ell^n\mathbb{Z})\times_{K/K0} S(K_0))(\mathbb{C})$, and if we tensorize with $\mathbb{Q}$, we get the sheaf over $S(K)(\mathbb{Q})$ defined by $V(\mathbb{Q})\times_{G(\mathbb{Q}),\xi} G(\mathbb{A})/K_\infty K$, this sheaf of $\mathbb{Q}$-vectorspaces over $S(K)(\mathbb{C})$ is denoted by $F_\xi(K)$.

1.2 Let $\mathcal{L}$, $\mathcal{W}$ and $\mathcal{D}$ be the gerbs (over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Q}_p$) constructed in LR - thus $\mathcal{W}$ is $G_m(\mathbb{C}) \to W_{\mathbb{R}} \to Gal(\mathbb{C}/\mathbb{R})$, for $\mathcal{L}$ and $\mathcal{D}$ see the appendix. And let, for ℓ prime and $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$, G_ℓ be the trivial gerb over $\mathbb{Q}_\ell$, - that is $1 \to \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \to \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Let G resp. G_{ab} be the neutral gerb (over $\mathbb{Q}$) associated to G resp. $G_{ab} = G/G_{der}$ - thus G is $G(\overline{\mathbb{Q}}) \to G(\overline{\mathbb{Q}})\times\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let $\zeta_\infty: \mathcal{W}_\infty \to \mathcal{L}$, $\zeta_p: \mathcal{D}_p \to \mathcal{L}$ and, for $\ell \neq p$, $\zeta_\ell: G_\ell \to \mathcal{L}$ be the (local) homomorphisms of gerbs constructed in LR (see appendix). In order to define ζ_p resp. ζ_ℓ an imbedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ resp. $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ is needed. The first is one for which the induced place of E ($\subset \overline{\mathbb{Q}}$) is that given by the chosen prime ideal $\mathfrak{p}$, the second is arbitrary.

To X_∞ is associated an equivalence class of homomorphisms $\xi_\infty: \mathcal{W} \to G$: we define the homomorphism $\xi^0_\infty: \mathcal{W} \to G_E$ by w: $G_m(\mathbb{C}) \to \underline{S}(\mathbb{C})$ $(z \to (z, \bar{z}))$ on the kernel and $\tau \to (-1, 1)\iota$ (recall that $W_\mathbb{R}$ is generated by $\mathbb{C}^x$ and a τ such that $\tau^2 = -1$ and $\tau z = \bar{z}\tau$, ι is the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$) and choose $h \in X_\infty$ and let ξ_∞ be the composite $\mathcal{W} \to^{\xi 0\infty} G_S \to^h G_{G\mathbb{R}}$. It is trivial that the equivalence class of ξ_∞ is independent of the choice of h. We choose one of these ξ_∞.

For each prime number ℓ we have a canonical neutralization $\zeta_\ell: G_\ell \to G$.

If we compose an element $\mu: G_m \to G_\mathbb{C}$ of $\overline{M}_\mathbb{C}$ with $G \to G_{ab}$, then we get a coweight $\mu_{ab} \in X_*(G_{ab})$ which is independent of μ. To μ_{ab} we can associate a homomorphism $\psi_{\mu ab}: \mathcal{L} \to G_{ab}$ (see LR, p. 144 or appendix).

A homomorphism $\varphi: \mathcal{L} \to G$ is called *permissible* if

1) $\mathcal{L} \to^\varphi G \to G_{ab}$ is equivalent to $\psi_{\mu ab}$ (global condition)
2) $\varphi\circ\zeta_\infty$ is equivalent to ξ_∞ (local condition at ∞)
3) the set X_p constructed below is not empty (local condition at p)

4) for $\ell \neq p$ (and for an arbitrary imbedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$) is $\varphi\circ\zeta_\ell$ equivalent to ξ_ℓ (local condition at $\ell \neq p$).

Let φ: $\mathcal{L} \to \mathcal{G}$ be an arbitrary homomorphism. We assume in the rest of this paper that E *is unramified at* p. Let $\mathbb{Q}_p^{un}$ be a maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ containing $E_{\wp}$. $\xi_p = \varphi\circ\zeta_p$: $\mathcal{D} \to \mathcal{G}$ factorizes through $\mathcal{D} \to \mathcal{D}^L$ for some unramified extension L of $\mathbb{Q}_p$ (in $\mathbb{Q}_p^{un}$) (LR, p. 120). Thus we have a homomorphism of gerbs for some finite Galois extension L_1 of $\mathbb{Q}_p$:

$$\begin{array}{ccccc} L_1^\times & \to & \mathcal{D}^L{}_{L1} & \to & \mathrm{Gal}(L_1/\mathbb{Q}_p) \\ \downarrow & & \downarrow\xi_p & & \downarrow \\ G(L_1) & \to & G(L_1)\times\mathrm{Gal}(L_1/\mathbb{Q}_p) & \to & \mathrm{Gal}(L_1/\mathbb{Q}_p). \end{array}$$

As shown in LR, p. 167, we can, by enlarging L and replacing ξ_p by an equivalent, say $\xi_p' = \mathrm{ad}(v)\circ\xi_p$ for $v \in G(\overline{\mathbb{Q}}_p)$, assume that $L_1 = L$, so that we have a homomorphim of gerbs:

$$\begin{array}{ccccc} L^\times & \to & W_{L/\mathbb{Q}p} & \to & \mathrm{Gal}(L/\mathbb{Q}_p) \\ \downarrow & & \downarrow\xi_p' & & \downarrow \\ G(L) & \to & G(L)\times\mathrm{Gal}(L/\mathbb{Q}_p) & \to & \mathrm{Gal}(L/\mathbb{Q}_p), \end{array}$$

for some unramified extension L of $\mathbb{Q}_p$.

Let κ denote the completion of $\mathbb{Q}_p^{un}$. ξ_p' determines a homomorphism ξ: $W_{L/\mathbb{Q}p} \to G(\kappa)\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ (via the canonical homomorphism $W_{L/\mathbb{Q}p} \to \mathrm{Gal}(L/\mathbb{Q}_p)$). Choose a $w \in W_{L/\mathbb{Q}p}$ which is mapped to the Frobenius σ of $\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ and define $F \in G(\kappa)\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ and $b \in G(\kappa)$ by $F = b\times\sigma = \xi(w)$. $G(\kappa)\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ acts on the Tits building $B(G, \kappa)$.

We assume now that K *has the form* $K = K_p\cdot K^p$*, where* K_p *is hyperspecial*, that is, the stabelizer in $G(\mathbb{Q}_p)$ of a hyperspecial point x_0 of $B(G, \kappa)$ (see Ti). Then $G_{\mathbb{Q}p}$ is split over some unramified extension of $\mathbb{Q}_p$, we assume that $G_{\mathbb{Q}p}$ *is quasi-split*. K_p is the set of points with coordinates in $\mathbb{Z}_p$ of a scheme defined over $\mathbb{Z}_p$, this scheme is also denoted K_p. If we base change with $\mathbb{Z}_p \to \mathbb{Q}_p$, we get $G_{\mathbb{Q}p}$. K^p is as usual a compact open subgroup of $G(\mathbb{A}^p{}_f)$.

Let χ denote $G(\kappa)\cdot x_0$ and let X_p denote $\{x \in \chi \mid inv(x, Fx)) = \overline{M}_{\wp}\}$, here inv is defined by

$$\{G(\kappa) \text{ orbits in } \chi\times\chi\} \leftrightarrow K_p(O_\kappa)\backslash G(\kappa)/K_p(O_\kappa)$$
$$\leftrightarrow X_*(S)/\Omega(G(\kappa), S(\kappa)) \leftrightarrow \mathcal{M}(\kappa),$$

where S is a maximal κ-split torus of G_κ and $\overline{M}_{\wp}$ is the class in $\mathcal{M}(E_{\wp})$ corresponding to $\overline{M}_{\overline{\mathbb{Q}}}$ in $\mathcal{M}(\overline{\mathbb{Q}}_p)$ ($\overline{M}_{\overline{\mathbb{Q}}}$ is fixed by $Gal(\overline{\mathbb{Q}}/E_{\wp})$ - for all this, see K4). As mentioned, X_p shall be nonempty in order for φ being permissible.

Let $\varphi: \mathcal{L} \to G$ be permissible. We introduce the notation:

$X_\ell = \{x \in G(\overline{\mathbb{Q}}_\ell) \mid \varphi\circ\zeta = ad(x)\circ\xi_\ell\}$ for $\ell \neq p$
$X^p = \Pi_{\ell\neq p} X_\ell$ (restricted product, see LR. p. 168)
$I_\varphi = \{g \in G(\overline{\mathbb{Q}}_p) \mid ad(g)\circ\varphi = \varphi\}$
$J_\varphi = \{g \in G(\overline{\mathbb{Q}}_p) \mid ad(g)\circ\xi_p = \xi_p\}$
$J_\varphi' = \{g \in G(\kappa) \mid ad(g)\circ\xi_p' = \xi_p'\}$

$G(\mathbb{Q}_\ell)$ acts simply transitively on X_ℓ (from right), therefore $G(\mathbb{A}^p{}_f)$ acts simply transitively on X^p. I_φ acts on X_ℓ (from left) and so on X^p. ad(v) induces a bijection $J_\varphi \leftrightarrow J_\varphi'$. J_φ', and therefore also J_φ, acts on X_p (from left), and because $I_\varphi \subset J_\varphi$, I_φ acts on X_p. Let $X_\varphi(K)$ denote the set $I_\varphi\backslash(X_p\times X^p/K^p)$, this set is non-empty because φ is permissible.

Let $\Phi = \Phi_{\wp}$ denote the element F^r in $G(\kappa)\times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$ (recall that $r = [E_{\wp}:\mathbb{Q}_p]$). Then Φ acts on X_p and therefore also on $X_\varphi(K)$.

We assume that

$$S_{\wp}(K)(\bar{\kappa}) = \sqcup_{\{\varphi\}} X_\varphi(K),$$

where the disjoint union is taken over all equivalence classes of permissible homomorphisms $\varphi: \mathcal{L} \to G$, and we assume that the action of the Frobenius on $S_{\wp}(K)(\bar{\kappa})$ corresponds to the action of Φ on $X_\varphi(K)$ (see 3.1).

For $\varepsilon \in I_\varphi$ and $j \in \mathbb{N}$ we introduce the notation:

$Y^j{}_p = \{x \in X_p \mid \varepsilon' x = \Phi^j x\}$
$Y^p = \{yK^p \in X^p/K^p \mid y^{-1}\varepsilon y \in K^p\}$,

here ε' for $\varepsilon \in I_\varphi$ denotes the element $ad(v)(\varepsilon)$ ($\in J_\varphi'$). We have an action of $(I_\varphi)_\varepsilon$ on $Y^j{}_p x Y^p$ (via $(I_\varphi)_\varepsilon \subset (J_\varphi)_\varepsilon$ and $ad(v): (J_\varphi)_\varepsilon \leftrightarrow (J_\varphi')_{\varepsilon'}$). The set $(I_\varphi)_\varepsilon \backslash (Y_p{}^j \times Y^p)$ is finite.

Let $\sim_K$ be the equivalence relation "conjugation modulo $Z(\mathbb{Q})_K$" on I_φ ($Z(\mathbb{Q})_K = Z(\mathbb{Q}) \cap K$). Then we have a map

$$X_\varphi(K)^{\Phi j} \to I_\varphi/\sim_K,$$

given by: if $\{(x_p, x^p)\} \in X_\varphi(K)^{\Phi j}$, then $\varepsilon' x_p = \Phi^j x_p$ and $\varepsilon x^p = x^p$ for some $\varepsilon \in I_\varphi$, let $\{(x_p, x^p)\}$ maps to $\{\varepsilon\}$.

We can choose K^p so small that

1) if $\varepsilon \in I_\varphi$ has a fixed point in $X_p \times (X^p/K^p)$, then $\varepsilon \in Z(\mathbb{Q})_K$

2) if $\varepsilon, \bar{\varepsilon} \in I_\varphi$ and $z \in Z(\mathbb{Q})_K$ and $\varepsilon^{-1}\bar{\varepsilon}\varepsilon = \bar{\varepsilon} z$, then $z = 1$.

Then for $\varepsilon \in I_\varphi$, the inverse image of $\{\varepsilon\}$ by the above map is $(I_\varphi)_\varepsilon \backslash (Y^j{}_p \times Y^p)$.

1.3 Let $\varphi: \mathcal{L} \to G$ be permissible, let $\varepsilon \in I_\varphi$ and let $j \in \mathbb{N}$, then, if $(I_\varphi)_\varepsilon \backslash (Y^j{}_p \times Y^p)$ is non-empty:

1) $\exists x \in G(\kappa)\cdot x_0$: $\varepsilon' x = \Phi^j x$

2) $\exists y \in X^p$: $y^{-1}\varepsilon y \in G(\mathbb{A}^p_f)$

We will call the pair (φ, ε) *j-permissible* if these two conditions are satisfied. If $\bar{\varphi} = ad(g)\circ\varphi$ and $\bar{\varepsilon} = {}^g\varepsilon$ resp. ${}^g\varepsilon\cdot z$ for $g \in G(\bar{\mathbb{Q}})$ and $z \in Z(\mathbb{Q})_K$, then $(\bar{\varphi}, \bar{\varepsilon})$ is also j-permissible - in this case $(\bar{\varphi}, \bar{\varepsilon})$ and (φ, ε) are called equivalent resp. K-equivalent.

For $n \in \mathbb{N}$, let F^n be the extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^{un}$ of degree n.

Because of 1) $\varepsilon'^{-1}\Phi^j$ has a fixed point in $G(\kappa)\cdot x_0$, therefore there exists a $c \in G(\kappa)$ such that $c(\varepsilon'^{-1}\Phi^j)c^{-1} = \sigma^n$ (K4, p. 291). Define $\delta \in G(\kappa)$ by $\delta = cb\sigma(c)^{-1}$ (recall that $\Phi = (b\times\sigma)^r$), then $\delta \in G(F^n)$ ($n = jr$) and $Nm_{F^n/\mathbb{Q}p}\delta = c\varepsilon' c^{-1}$. The σ-conjugacy class of δ in $G(F^n)$ is determined by the equivalence class of (φ, ε).

Because of 2) $\gamma = y^{-1}\varepsilon y$ belongs to $G(\mathbb{A}^p_f)$. The conjugacy class of γ in $G(\mathbb{A}^p_f)$ is determined by the equivalence class of (φ, ε).

For $n = jr$, let $f_{\wp,n} \in \mathscr{H}(G(F^n), K_p(\mathcal{O}_{F^n}))$ be meas $(K_p(\mathcal{O}_{F^n})/(Z_K)_p)^{-1}$ · the characteristic function of the coset in $K_p(\mathcal{O}_{F^n})\backslash G(F^n)/K_p(\mathcal{O}_{F^n})$ corresponding to $\overline{M}_\wp \in \mathscr{H}(F^n)$ ($(Z_K)_p = Z(\mathbb{Q}_p)\cap K_p$). Let $\varphi^p \in \mathscr{H}(G(\mathbb{A}^p_f), K^p)$ be meas$(K^p/(Z_K)^p)^{-1}$ · the characteristic function of K^p ($(Z_K)^p = Z(\mathbb{A}^p_f)\cap K^p$).

Let $G^\sigma_\delta(\mathbb{Q}_p)$ denote the σ-centralizer of δ in $G(F^n)$, that is, $\{g \in G(F^n) \mid g^{-1}\delta\sigma(g) = \delta\}$, this subgroup is defined over $\mathbb{Q}_p$ (if $G^\sim = Res_{F^n/\mathbb{Q}p}G$ and θ is the $\mathbb{Q}_p$-automorphism of $G^\sim$ corresponding to the action of σ on $G(F^n) = G^\sim(\mathbb{Q}_p)$, then G^σ_δ is the set of fixed points of $ad(\delta)\circ\theta$).

The following computation of $|(I_\varphi)_\varepsilon\backslash(Y^j_p\times Y^p)|$ is the principal idea of K4.

We have bijections

$$Y^j_p \leftrightarrow \{g_pK_p(\mathcal{O}_F{}^n) \in G(F^n)/K_p(\mathcal{O}_F{}^n) \mid f\tilde{}_{\wp,n}((g_p)^{-1}\delta\sigma(g_p) \neq 0\}$$
$$(g_px_0 \rightarrow cg_pK_p(\mathcal{O}_F{}^n))$$

and

$$Y^p \leftrightarrow \{g^pK^p \in G(\mathbb{A}^p{}_f)/K^p \mid \varphi^p((g^p)^{-1}\gamma g^p) \neq 0\}$$
$$(yg^pK^p \leftarrow g^pK^p).$$

With use of these we get

$$|(I_\varphi)_\varepsilon\backslash(Y^j{}_p\times Y^p)|$$
$$= \mathrm{meas}(K_p(\mathcal{O}_F{}^n)/(Z_K)_p)\cdot \mathrm{meas}(K^p/(Z_K)^p)$$
$$\Sigma\, f\tilde{}_{\wp,n}((g_p)^{-1}\delta\sigma(g_p))\ \varphi^p((g^p)^{-1}\gamma g^p)$$
$$(\text{sum: } \{(g_p, g^p)\} \in (I_\varphi)_\varepsilon\backslash(G(F^n)\times G(\mathbb{A}^p{}_f))/K_p(\mathcal{O}_F{}^n)\times K^p)$$
$$= \int f\tilde{}_{\wp,n}((g_p)^{-1}\delta\sigma(g_p))\ \varphi^p((g^p)^{-1}\gamma g^p)\ dg_pdg^p/dh$$
$$(\text{integral: } (I_\varphi)_\varepsilon Z_K\backslash(G(F^n)\times G(\mathbb{A}^p{}_f)))$$
$$= \mathrm{meas}((I_\varphi)_\varepsilon Z_K\backslash(G^\sigma{}_\delta(\mathbb{Q}_p)\times G_\gamma(\mathbb{A}_f{}^p)))\cdot TO(\delta, f\tilde{}_{\wp,n})\cdot O(\gamma, \varphi^p).$$

Here $(I_\varphi)_\varepsilon$ acts on $G(F^n)$ and $G(\mathbb{A}^p{}_f)$ via the imbeddings $ad(cv)$: $(I_\varphi)_\varepsilon \rightarrow G^\sigma{}_\delta(\mathbb{Q}_p)$ and $ad(y^{-1})$: $(I_\varphi)_\varepsilon \rightarrow G_\gamma(\mathbb{A}^p{}_f)$, we identify $(I_\varphi)_\varepsilon$ with its image in $G(F^n)\times G(\mathbb{A}^p{}_f)$. $(I_\varphi)_\varepsilon Z_K$ is closed in $G(F^n)\times G(\mathbb{A}_f{}^p)$ and the intersection of $(I_\varphi)_\varepsilon Z_K$ with any conjugate of $K_p(\mathcal{O}_F{}^n)\cdot K^p$ is equal to Z_K (this follows from condition 1) of K^p in l.2). $TO(\delta, f)$ is the twisted orbital integral of the function f on $G(F^n)$ at $\delta \in G(F^n)$ and $O(\gamma, \varphi)$ is the orbital integral of the function φ on $G(\mathbb{A}_f{}^p)$ at $\gamma \in G(\mathbb{A}^p{}_f)$. The measures on $G(F^n)$, $G(\mathbb{A}_f{}^p)$ and $(I_\varphi)_\varepsilon Z_K$ are arbitrary, and the measures on the compact open subgroups resp. $K_p(\mathcal{O}_F{}^n)$, K^p and Z_K are the restrictions the measures on $G^\sigma{}_\delta(\mathbb{Q}_p)$ and $G_\gamma(\mathbb{A}^p{}_f)$ are also arbitrary.

1.4 Let ${}^L G^0$ denote the connected L-group of G. It is provided with a Cartan subgroup ${}^L T^0$, a Borel subgroup ${}^L B^0$, an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ leaving these subgroups invariant, and for a Cartan subgroup T of G we can choose an isomorphism $X_*(T) \leftrightarrow X^*({}^L T^0)$ (determined up to composition with a Weyl-group action). Let Z denote the center of ${}^L G^0$. Z is connected because G_{der} is simply connected.

The class $\overline{M}_{\mathbb{C}}$ determines a Weyl-group orbit Ω_μ in X^* $({}^L T^0)$. The restrictions of the characters in Ω_μ to Z is one and the same character and is denoted by μ_2.

Recall that G is assumed to be unramified over $\mathbb{Q}_p$, that is, quasi-split over $\mathbb{Q}_p$ and split over some unramified extension of $\mathbb{Q}_p$.

Let, for $\varepsilon \in G(\mathbb{Q}_p)_{s.s.}$ (s.s. = semi-simple), M_ε denotes the centralizer in $G_{\mathbb{Q}p}$ of the maximal $\mathbb{Q}_p$-split torus in the center of $(G_{\mathbb{Q}p})_\varepsilon$.

Let, for $j \in \mathbb{N}$, $G(\mathbb{Q}_p)^n$ ($n = jr$) denote the set of elements ε in $G(\mathbb{Q}_p)_{s.s.}$ such that:

there exist a Cartan subgroup T of M_ε and a $\mu \in X_*(T)$ such that:

1) μ is defined over F^n

2) the class in $\mathscr{M}(F^n)$ containing μ is $\overline{M}_{\wp}$

3) if $Z_{M\varepsilon}$ is the center of the connected L-group ${}^L M^0_\varepsilon$ of M_ε and if $\mu_1 \in X^*(Z_{M\varepsilon})$ is the restriction of μ (via the Cartan subgroup ${}^L T^0_{M\varepsilon}$ and an isomorphism $X_*(T) \leftrightarrow X({}^L M^0_\varepsilon)$ used in the construction of ${}^L M^0_\varepsilon$), then $Nm_{F^n/\mathbb{Q}p}\mu_1$ is the image of ε by the map λ: $M_\varepsilon(\mathbb{Q}_p) \to X^*(Z)^{Gal(\overline{\mathbb{Q}}p/\mathbb{Q}p)}$ constructed in K4, p. 298 (M_ε is split over an unramified extension of $\mathbb{Q}_p$ since $G_{\mathbb{Q}p}$ is).

In G resp. $G_{\mathbb{Q}\nu}$ (ν place) stable conjugacy is the same as $G(\overline{\mathbb{Q}})$- resp. $G(\mathbb{Q}_\nu)$-conjugacy (because G_{der} is simply connected).

If $\varepsilon \in G(\mathbb{Q}_p)^n$ and $\varepsilon' \in G(\overline{\mathbb{Q}}_p)$ is stably conjugate to ε

(modulo $Z(K)_p$), then $\varepsilon' \in G(\mathbb{Q}_p)^n$ (LR, Lemma 5.17).

Let $G(\mathbb{Q})^n_\infty$ denote $\{g \in G(\mathbb{Q}_p)_{s.s.} \mid g \in G(\mathbb{Q}_p)^n$ and g is elliptic at infinity$\}$.

Let $\sim_K$ denote the equivalence relation "$G(\mathbb{Q})$-conjugation modulo $Z(\mathbb{Q})_K$" on $G(\overline{\mathbb{Q}})$.

If $\varepsilon \in G(\mathbb{Q})^n_\infty$ and $\varepsilon' \in G(\mathbb{Q})$ and $\varepsilon \sim_K \varepsilon'$ (that is, ε' and ε are stably conjugate modulo $Z(\mathbb{Q})_K$), then $\varepsilon' \in G(\mathbb{Q})^n_\infty$.

We now assume that *the Hasse princip is true for* G_{der} (this is true if G_{der} has no E_8 factor). Then if (φ, ε) is a j-permissible pair, there exists a $\varepsilon' \in G(\mathbb{Q})$ such that $\varepsilon \sim_K \varepsilon'$, and a such ε' belongs to $G(\mathbb{Q})^n_\infty$, and, conversely, if $\varepsilon' \in G(\mathbb{Q})^n_\infty$, there exists a j-permissible pair (φ, ε) such that $\varepsilon \sim_K \varepsilon'$ (LR, Satz 5.21).

1.5 Let, for $\varepsilon \in G(\mathbb{Q})^n_\infty$, P_ε denote the set of G_ε-equivalence classes of permissible homomorphisms $\varphi: \mathscr{L} \to G$ such that (φ, ε) is j-permissible. Then $P_\varepsilon \neq \emptyset$ and every j-permissible pair $(\overline{\varphi}, \overline{\varepsilon})$ is equivalent to a pair (φ, ε), where $\varepsilon \in G(\mathbb{Q})^n_\infty$ and $\varphi \in P_\varepsilon$.

For $\varepsilon \in G(\mathbb{Q})^n_\infty$ there exist a $\overline{\varepsilon} \in G(\mathbb{Q})^n_\infty$ such that $\overline{\varepsilon} \sim_K \varepsilon$ and such that for $\varphi \in P_{\overline{\varepsilon}}$ is the following condition satisfied: if L ($\subset \overline{\mathbb{Q}}$) is a Galois extension of $\mathbb{Q}$ and $m \in \mathbb{N}$, both chosen so large that φ factorizes through $\mathscr{L}^L{}_m$, then $\varphi(\delta_{\overline{m}}) \in G(\mathbb{Q})$ for $\overline{m}$ divisibel by m and sufficiently large (for the notation see appendix). A such $\overline{\varepsilon}$ is called *favourable*. In fact, if ε is favourable, then for every $\varphi \in P_\varepsilon$ the restriction of φ to the kernel of $\mathscr{L}$ is independent of φ and it maps into the center of G_ε (LR, p. l90 and p. 194). We choose a set of favourable representatives in the $\sim_K$-equvivalence classes of $G(\mathbb{Q})^n_\infty$, thus every j-permissible pair $(\overline{\varphi}, \overline{\varepsilon})$ is K-equivalent to a j-permisible pair (φ, ε) where ε is a such representative and $\varphi \in P_\varepsilon$.

For $\varphi \in P_\varepsilon$ (ε favourable), let $\mathcal{I}$ denote $G_\varphi(\delta_m)$ (m sufficiently large - $\mathcal{I}$ is independent af m because G_{der} is simply connected), let $\mathcal{I}_\varphi$ denote the inner twisting of $\mathcal{I}$ by φ (if $\varphi(t_\delta) = s_\sigma \times \sigma$ ($s_\sigma \in G_\varepsilon(\overline{\mathbb{Q}})$, $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ then $\sigma \to ad(s_\sigma)$ is a cocycle in $Aut(\mathcal{I}(\overline{\mathbb{Q}}))$ because $\varphi(Q(L, m)) \subset$ center $\mathcal{I}$), and also let $\mathfrak{J}$ denote the centralizer in $G_{\mathbb{Q}p}$ of the image of the kernel of $\mathcal{D}$ by $\xi_p = \varphi \circ \zeta_p$, and let $\mathfrak{J}_\varphi$ denote the inner twisting of $\mathfrak{J}$ by ξ_p. Then $\mathcal{I}_\varphi(\mathbb{Q}) = I_\varphi$ and $\mathfrak{J}_\varphi(\mathbb{Q}_p) = J_\varphi$, and $\mathcal{I}_{\mathbb{Q}p} \subset \mathfrak{J}$ and $(\mathcal{I}_\varphi)_{\mathbb{Q}p} \subset \mathfrak{J}_\varphi$. Moreover $G_\varepsilon \subset \mathfrak{J}$, and if $(G_\varepsilon)_\varphi$ denote the inner twisting of G_ε by φ, then $(G_\varepsilon)_\varphi \subset \mathfrak{J}_\varphi$. $(\mathfrak{J}_\varphi)_\mathbb{R}$ (and $((G_\varepsilon)_\varphi)_\mathbb{R}$) is independent of $\varphi \in P_\varepsilon$, in fact, ξ_∞ (see 1.2) defines an inner twisting $G_\mathbb{R}'$ of $G_\mathbb{R}$ (because it maps the kernel of $\mathcal{W}$ into the center of G), $Z(\mathbb{R})\backslash G_\mathbb{R}'(\mathbb{R})$ is compact (LR, p. 165) and $(\mathfrak{J}_\varphi)_\mathbb{R}$ (and $(G_\varepsilon)_\varphi)_\mathbb{R}$) is a subgroup af $G_\mathbb{R}'$.

For $\varphi \in P_\varepsilon$, let $v \in G(\overline{\mathbb{Q}}_p)$, $c \in G(\kappa)$, $\delta \in G(F^n)$, $y \in G(\mathbb{A}^p{}_f)$ and $\gamma \in G(\mathbb{A}^p{}_f)$ be as in 1.3. Then we have an isomorphism $((\mathfrak{J}_\varphi)_\varepsilon)(\mathbb{Q}_p) \leftrightarrow G^\sigma{}_\delta(\mathbb{Q}_p)$ given by ad(cv), and an isomorphism $(G_\varepsilon)_\varphi(\mathbb{A}^p{}_f) \leftrightarrow G_\gamma(\mathbb{A}^p{}_f)$ given by $ad(y^{-1})$. Therefore we have

$$\begin{aligned}
&\mathrm{meas}((I_\varphi)_\varepsilon Z_K\backslash(G^\sigma{}_\delta(\mathbb{Q}_p)\times G(\mathbb{A}^p{}_f))\\
&= \mathrm{meas}((\mathfrak{J}_\varphi)_\varepsilon(\mathbb{Q})Z_K\backslash(\mathfrak{J}_\varphi)_\varepsilon(\mathbb{A}_f))\\
&= \mathrm{meas}((G_\varepsilon)_\varphi(\mathbb{Q})Z_K\backslash(G_\varepsilon)_\varphi(\mathbb{A}_f))
\end{aligned}$$

(because $((\mathfrak{J}_\varphi)_\varepsilon)_{\mathbb{Q}p} = (\mathfrak{J}_\varphi)_\varepsilon$ - see K4).

For a reductive connected algebraic group G, the sign c(G) is defined in K2. We introduce the following abbreviations

$$c_\infty = c(((G_\varepsilon)_\varphi)_\mathbb{R}) = c(((G_\varepsilon)_\mathbb{R}'))$$
$$c_p = c(((G_\varepsilon)_\varphi))_{\mathbb{Q}p}) = c(G^\sigma{}_\delta)$$
$$c^p = c(((G_\varepsilon)_\varphi))_{\mathbb{A}^p{}_f}) = c(G_\gamma),$$

then $c_\infty c_p c^p = 1$.

1.6 We choose a measure on $Z(\mathbb{R})$, and for each $\varepsilon \in G(\mathbb{R})_e$ (e = elliptic) we choose a measure on $G_\varepsilon(\mathbb{R})$ such that if ε' and ε are stably conjugate (and therefore $G_{\varepsilon'}$ is an inner form of G_ε) the measures on $G_{\varepsilon'}(\mathbb{R})$ and $G_\varepsilon(\mathbb{R})$ are compatible. Then we have a measure on $G_\varepsilon'(\mathbb{R})$ for each $\varepsilon \in G(\mathbb{R})_e$ (recall that $Z(\mathbb{R})\backslash G_\varepsilon'(\mathbb{R})$ is compact).

We define a function $\alpha: G(\mathbb{R}) \to \mathbb{R}$ by

$$\alpha(\varepsilon) = c(G_\varepsilon') \operatorname{tr} \xi(\varepsilon)/\operatorname{meas}(Z(\mathbb{R})\backslash G_\varepsilon'(\mathbb{R}))$$
$$\text{if } \varepsilon \in G(\mathbb{R})_e \text{ and } 0 \text{ if } \varepsilon \in G(\mathbb{R})\backslash G(\mathbb{R})_e.$$

Let, for a reductive $\mathbb{Q}$-group $\overline{G}$ in whose center the center Z of G can be canonically imbedded and for which $\overline{G}_\mathbb{R}$ has an inner form $\overline{G}_\mathbb{R}'$ such that $Z(\mathbb{R})\backslash\overline{G}_\varepsilon'(\mathbb{R})$ is compact, $\tau(\overline{G})_K$ denote $\operatorname{meas}(\overline{G}(\mathbb{Q})Z(\mathbb{R})Z_K\backslash\overline{G}(\mathbb{A}))$. Then we have for $\varepsilon \in G(\mathbb{Q})^n{}_\infty$ and $\varphi \in P_\varepsilon$:

$$\tau(\overline{G})_K = \operatorname{meas}(Z(\mathbb{R})\backslash(G_\varepsilon)_\mathbb{R}'(\mathbb{R}))$$
$$\cdot\operatorname{meas}((G_\varepsilon)_\varphi(\mathbb{Q})Z_K\backslash(G_\varepsilon)_\varphi(\mathbb{A}_f))$$

(recall that the measure on $(G_\varepsilon)_\varphi(\mathbb{A}_f)$ is defined by the isomorphism $(G_\varepsilon)_\varphi(\mathbb{A}_f) \leftrightarrow (G^\sigma{}_\delta)(\mathbb{Q}_p)\times G_\gamma(\mathbb{A}^p{}_f))$.

1.7 In this section we let Γ and Γ_v (v place) denote resp. $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\operatorname{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$.

If $\overline{G}$ is a connected reductive $\mathbb{Q}_p$-group, we denote by $B(\overline{G})$ the group $\overline{G}(\kappa)/\sim$, where $\sim$ is the equivalence relation "σ-conjugation" (that is, $g' \sim g'' \Leftrightarrow \exists g \in \overline{G}(\kappa)$: $g' = gg''\sigma(g)^{-1}$ (σ the Frobenius of $\operatorname{Gal}(\kappa/\mathbb{Q}_p)$), and by $B(\overline{G})_b$ the subgroup of basic elements (see K5). Then we have an isomorphism $B(\overline{G})_b \leftrightarrow X^*(\overline{Z}^{\Gamma p})$, where $\overline{Z}$ is the center of the connected L-group of $\overline{G}$, and a homomorphism $B(\overline{G})_b \to X^*(\overline{Z})^{\Gamma p}\otimes\mathbb{Q}$ with kernel $H^1(\mathbb{Q}_p, \overline{G}) = \pi_0(\overline{Z}^{\Gamma p})^D$.

For j $\in \mathbb{N}$ and $\varepsilon \in G(\mathbb{Q}_p)^n$ we introduce the notation:

$\Psi^n_\varepsilon = \{\delta \in G(F^n) \mid \exists c \in G(\kappa)\colon Nm_{F^n/\mathbb{Q}_p}\delta = c\varepsilon c^{-1} \wedge \delta$ is mapped by $G(\kappa) \to G_{ab}(\kappa) \to B(G_{ab}) \to X^*(Z^{\Gamma p})$ to the restriction of $\mu_2 \in X^*(Z)\}/\sim$,

$\Phi^n_\varepsilon = \{b \in G_\varepsilon(\kappa) \mid \exists c \in G(\kappa)\colon Nm_{F^n/\mathbb{Q}_p}b = \varepsilon(c^{-1}\sigma^n(c)) \wedge b$ is mapped by $G(\kappa) \to G_{ab}(\kappa) \to B(G_{ab}) \to X^*(Z^{\Gamma p})$ to the restriction of $\mu_2 \in X^*(Z)\}/\sim$

and, if $b_0 \in \Phi^n_\varepsilon$, then a conjugation on $G_\varepsilon(\kappa)$ is defined by $g \to \sigma'(g) = b_0\sigma(g)b_0^{-1}$ (because $Nm_{F^m/\mathbb{Q}_p}b_0 \in$ center $G_\varepsilon(\kappa)$ for m suffenciently large) and we let

$\Phi_\varepsilon' = \{a \in G_\varepsilon(\kappa) \mid \exists n' \in \mathbb{N}, b \in G_\varepsilon(\kappa)\colon Nm'_{F^{n'}/\mathbb{Q}_p}a = b^{-1}\sigma'^{n'}(b) \wedge a$ is mapped by $G(\kappa) \to G_{ab}(\kappa) \to B(G_{ab})$ to the identity$\}/\sim'$

(here Nm' is the norm associated to σ' and ~ resp. ~' is the equivalence relation "σ-conjugation" resp. "σ'-conjugation").

Let G_ε' denote the inner twisting of G_ε determined by σ', let maps

$\varphi\colon B(G_\varepsilon)_b \to X^*(Z_\varepsilon)^{\Gamma p}\otimes\mathbb{Q}$

and

$\varphi'\colon B(G_\varepsilon')_b \to X^*(Z_\varepsilon)^{\Gamma p}\otimes\mathbb{Q}$

be as above (Z_ε is the center of the connected L-group of G_ε and G_ε'), and let

$\psi\colon B(G_\varepsilon)_b \to B(G_{ab})$

and

$\psi'\colon B(G_\varepsilon')_b \to B(G_{ab})$

be the projections. Then we have

$\Psi_\varepsilon' = \varphi'^{-1}(0)\cap\psi'^{-1}(0) = \ker(H^1(\mathbb{Q}_p, G_\varepsilon') \to H^1(\mathbb{Q}_p, G_{ab}))$

and

$\Psi^n_\varepsilon = \varphi^{-1}(\tau/n)\cap\psi^{-1}(\mu_2|Z^{\Gamma p})$

where $\tau = \lambda(\varepsilon)|(Z_{M\varepsilon})^{\Gamma p}$ (see 1.4), we have used that $X^*(Z_{M\varepsilon}{}^{\Gamma p})\otimes\mathbb{Q} = X^*(Z_\varepsilon)^{\Gamma p}\otimes\mathbb{Q}$ and $B(G_{ab}) = X^*(Z^{\Gamma p})$.

We have a bijection

$$\Phi_\varepsilon' \leftrightarrow \Phi^n{}_\varepsilon$$

given by $a \to ab_0$, and a bijection

$$\Phi^n{}_\varepsilon \leftrightarrow \Psi^n{}_\varepsilon$$

given by $b \to cb\sigma(c)^{-1}$. We identify $\Phi^n{}_\varepsilon$ and $\Psi^n{}_\varepsilon$.

For $\varepsilon \in G(\mathbb{Q}_p)^n$ we construct an element $b_\varepsilon \in \Phi^n{}_\varepsilon$ in the following way: we choose an elliptic Cartan subgroup T of G_ε, and a coweight $\mu \in X_*(T)$ which is M_ε -conjugate to a μ satisfying the condition in 1.4, then the homomorphism $\xi_{-\mu}: \mathcal{D} \to T(\overline{\mathbb{Q}}_p)\times\Gamma_p$ (see LR or appendix) is basic for G_ε, that is, if we by the procedure of 1.2 construct a homomorphism $\xi: W_{L/\mathbb{Q}p} \to T(\kappa)\times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$ for some unramified extension L of $\mathbb{Q}_p$ (in κ) and let $\xi(w) = b\times\sigma$, then b is basic in $G_\varepsilon(\kappa)$ (because T is elliptic in G_ε), and we take $b_\varepsilon = \{b\}$. The element in $X^*((Z_\varepsilon)^{\Gamma p})$ corresponding to b_ε is $\mu|(Z_\varepsilon)^{\Gamma p}$ (we have chosen an identification $X_*(T) \leftrightarrow X^*({}^LT^0{}_{G\varepsilon})$), b_ε can also be constructed as follows: choose (T, μ) as above, then we can choose a κ-split Cartan subgroup T' of M_ε such that the image of T' in $(M_\varepsilon)_{ad}$ is anisotropic and a $\mu \in X_*(T')$ such that μ' is M_ε-conjugate to μ, and now the homomorphism $\xi_{-\mu'}: \mathcal{D} \to T'(\overline{\mathbb{Q}}_p)\times\Gamma_p$ already has the wanted form (that is, it comes from a homomorphism $\xi_{-\mu}: W_{L/\mathbb{Q}p} \to T'(L)\times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$, where L splits T'), therefore the corresponding "b" is simply $\mu'(p)$ ($\in T'(L)$), and because $\xi_{-\mu}$ and $\xi_{-\mu'}$ are M_ε-conjugate, that is $\xi_{-\mu'} = ad(u)\circ\xi_{-\mu}$ for some $u \in M_\varepsilon(\overline{\mathbb{Q}}_p)$ (see LR, p. 172), we have $b = u'^{-1}\mu'(p)\sigma(u') \in G_\varepsilon(\kappa)$ if we write $u = u'v$ for $u' \in M_\varepsilon(\mathbb{Q}_p{}^{un})$ and $v \in G_\varepsilon(\overline{\mathbb{Q}}_p)$.

The last (by 3.2) and the last but one (obvious) equation of this paragraph are independent of the choise of b_ε, (i.e. of (T, μ)).

For $\rho \in \mathscr{E}(G_\varepsilon/\mathbb{Q}_p)$ we have a bijection

$$\theta^\rho: \Phi^n{}_\varepsilon \leftrightarrow \Phi^n{}_{\rho\varepsilon}$$

given by $\{b\} \rightarrow \{gb\sigma(b)^{-1}\}$ if ρ is given by $\sigma \rightarrow g^{-1}\sigma(g)$ $(\in G_\varepsilon(\overline{\mathbb{Q}}_p)$, $\sigma \in \Gamma_p)$ and g is chosen in $G(\mathbb{Q}_p{}^{un})$. And if we choose a pair (T, μ) as above, we have a bijection

$$h: \mathscr{E}(G_\varepsilon/\mathbb{Q}_p) \leftrightarrow \Phi^n{}_\varepsilon$$

given by $\rho \rightarrow (\theta^\rho)^{-1}(b_{\rho\varepsilon})$, here $b_{\rho\varepsilon}$ is defined by $({}^gT, {}^g\mu)$ (g $\in G(\overline{\mathbb{Q}}_p))$ chosen such that $\rho = \{g^{-1}\sigma(g)\} \in \mathscr{E}(T/\mathbb{Q}_p)$, it is possible because T is elliptic in G_ε). We have $h(\{g^{-1}\sigma(g)\})$ $= \{bg^{-1}\sigma(g)\}$ if $b_\varepsilon = \{b\}$.

Let $\varepsilon \in G(\mathbb{Q})^n{}_\infty$ and assume that ε is favourable. Choose $\varphi_0 \in P_\varepsilon$. Then φ_0 deternines a $b_0 \in \Phi^n{}_\varepsilon$ and a $\gamma_0 \in G(\mathbb{A}^p{}_f)$ (see 1.3) and a twisted form G_ε' of G_ε.

Let $K(G_\varepsilon/\mathbb{Q})$ denote the set of elements $\pi_0((Z_\varepsilon/Z)^\Gamma)$ for which the associated element in $H^1(\mathbb{Q}, Z)$ is locally trivial. $K(G_\varepsilon/\mathbb{Q})$ is a group and if we let X denote the group $(\pi_0((Z_\varepsilon/Z)^\Gamma))^D$ and, for every place ν, let X_ν denote the subgroup obtained by restricting to $\pi_0((Z_\varepsilon/Z)^\Gamma)$, the kernel of $\pi_0((Z_\varepsilon/Z)^{\Gamma\nu})^D \rightarrow \pi_0(Z^{\Gamma\nu}{}_\varepsilon)^D$, then

$$K(G_\varepsilon/\mathbb{Q})^D = X/\Pi_\nu X_\nu.$$

Because $\mathscr{E}(G_\varepsilon, \mathbb{Q}_\nu)$ (ν place) is the kernel of $(\pi_0(Z^{\Gamma\nu}{}_\varepsilon))^D \rightarrow (\pi_0(Z^{\Gamma\nu}))^D$, we easy see that we have a natural homomorphism

$$\mathscr{E}(G_\varepsilon/\mathbb{Q}_\nu) \rightarrow K(G_\varepsilon/\mathbb{Q})^D.$$

We can also construct a map

$$\Phi^n{}_\varepsilon \rightarrow K(G_\varepsilon/\mathbb{Q})^D$$

in the following way: choose a Cartan subgroup T of G_ε

which is elliptic in $G(\mathbb{R})$ and a $h \in X_\infty$ which factorizes through T (ε is elliptic in $G(\mathbb{R})$), then the restriction of μ_h to $(Z_\varepsilon)^{\Gamma\infty}$ is independent of the choices (LR, p. 184, Z_ε can be canonically imbedded in ${}^LT^0$ and we have identified $X_*(T)$ and $X^*({}^LT^0)$), and the restriction of μ_h to $Z^{\Gamma\infty}$ is $\mu_2|Z^{\Gamma\infty}$, therefore we can construct a character λ_∞ of $\ker(Z_\varepsilon \to (Z_\varepsilon/Z)^{\Gamma\infty})$ whose restriction to $(Z_\varepsilon)^{\Gamma\infty}$ is $\mu_h|(Z_\varepsilon)^{\Gamma\infty}$ and whose restriction to Z is μ_2, furthermore, if $b \in \Phi^n{}_\varepsilon$, then because $\Phi^n{}_\varepsilon \subset B(G_\varepsilon)_b$, a character μ_b of $(Z_\varepsilon)^{\Gamma p}$ is attached to b, and the restriction of μ_b to $Z^{\Gamma p}$ is $\mu_2|Z^{\Gamma\infty}$, therefore we can construct a character λ_p of $\ker(Z_\varepsilon \to (Z_\varepsilon/Z)^{\Gamma p})$ whose restriction to $Z^{\Gamma p}{}_\varepsilon$ is μ_b and whose restriction to Z is μ_2. Now, if λ_∞' and λ_p' are the restrictions of λ_∞ and λ_p to $\ker(Z_\varepsilon \to (Z_\varepsilon/Z)^\Gamma)$, then $\lambda_p'\cdot(\lambda_\infty')^{-1}$ is a character of $(Z_\varepsilon/Z)^\Gamma$, and this is trivial on the identity component, thus we have a character in $\pi_0(Z_\varepsilon/Z)^\Gamma)$ and so an element of $K(G_\varepsilon/\mathbb{Q})^D$ - this element is independent of the choises.

We consequently have a map

$$\beta\colon \Phi^n{}_\varepsilon\times\mathscr{E}(G_\varepsilon, \mathbb{A}^p{}_f) \to K(G_\varepsilon/\mathbb{Q})^D.$$

Also, we have a commutative diagram

$$\begin{array}{ccc} & (a, \rho) \quad \to \quad (ab_0, \rho\rho_0) & \\ & \Phi_\varepsilon'\times\mathscr{E}(G_{\gamma 0}, \mathbb{A}^p{}_f) \leftrightarrow \Phi^n{}_\varepsilon\times\mathscr{E}(G_\varepsilon/\mathbb{Q})^D & \\ & \uparrow \qquad\qquad\qquad \uparrow A & \\ \ker(H^1(\mathbb{Q}, G_\varepsilon') \to & (H^1(\mathbb{Q}, G_{ab})\times H^1(\mathbb{R}, G_\varepsilon')) \leftrightarrow P_\varepsilon, & \end{array}$$

the lower map maps the cocycle $c\colon \Gamma \to G_\varepsilon'(\mathbb{Q})$ to its product with φ_0 ($\in P_\varepsilon$), it is a bijection (LR, Lemma 5,26), the vertical maps are resp. the natural map and the map A given by $\varphi \to (\delta, \gamma)$ (see 1.3, recall that we have identified $\Psi^n{}_\varepsilon$ and $\Phi^n{}_\varepsilon$).

If we choose a Cartan subgroup T of G_ε (elliptic in $G(\mathbb{R})$ and in $G_\varepsilon(\mathbb{Q}^p)$), an $h \in X_\infty$ which factorizes through T and a $\mu \in X_*(T)$ which is M_ε-conjugate to a μ satisfying the condition in 1.4, then $\mu - \mu_h$ determines an element in $\mathscr{E}(T/\mathbb{R})$ (via the Tate-Nakayama isomorphism) and $\beta(b_\varepsilon)$ is equal to the image of that element in $K(G_\varepsilon/\mathbb{Q})^D$.

For $\kappa \in K(G_\varepsilon/\mathbb{Q})$ define G^κ: $\Psi^n{}_\varepsilon \times \mathscr{E}(G_\varepsilon, \mathbb{A}^p{}_f) \to \mathbb{C}$ by

$$G^\kappa(\delta, \rho) = \kappa(\beta(\delta, \rho))\cdot c(G^\sigma{}_\delta)\cdot TO(\delta, f^{\sim}{}_{\wp,n})\cdot c(G_{\delta\varepsilon})\cdot O({}^\delta\varepsilon, \varphi^p).$$

Then we have for $\varphi \in P_\varepsilon$

$$G^\kappa(A(\varphi)) = c(G^\sigma{}_\delta)\cdot TO(\delta, f^{\sim}{}_{\wp,n})\cdot c(G_\gamma)\cdot O(\gamma, \varphi^p)$$

(if $A(\varphi) = (\delta, \rho)$) for any $\kappa \in K(G_\varepsilon/\mathbb{Q})$, and

$\Sigma\, G^\kappa(\delta, \rho)$ (sum over $\kappa \in K(G_\varepsilon/\mathbb{Q})$) $= 0$ for $(\delta, \rho) \notin A(P_\varepsilon)$ (LR, Satz 5.25).

The number of elements in P_ε which by A are mapped to a given element in the image is always

$$i(\varepsilon) = |\ker(H^1(\mathbb{Q}, G_\varepsilon) \to H^1(\mathbb{Q}, G_{ab})\times H^1(\mathbb{A}, G_\varepsilon))|$$

- $i(\varepsilon') = i(\varepsilon)$ if $\varepsilon \sim_K \varepsilon'$ (LR, p. 193).

Now we can rewrite (6)

$$\Sigma\, c(G^\sigma{}_\delta)\cdot TO(\delta, f^{\sim}{}_{\wp,n})\cdot c(G_\gamma)\cdot O(\gamma, \varphi^p) \text{ (sum over } \varphi \in P_\varepsilon)$$

$$= i(\varepsilon)\cdot|K(G_\varepsilon/\mathbb{Q})|^{-1}\, \Sigma\, \Sigma\, G^\kappa(\delta, \rho)$$

(sum over $\kappa \in K(G_\varepsilon/\mathbb{Q})$, $(\delta, \rho) \in \Psi^n{}_\varepsilon \times \mathscr{E}(G_\varepsilon, \mathbb{A}^p{}_f)$)

$$= i(\varepsilon)\cdot|K(G_\varepsilon/\mathbb{Q})|^{-1}\, \Sigma\, \kappa_\infty(\mu - \mu_h) \cdot(\Sigma\, \kappa_p(\rho)\cdot c(G^\sigma{}_\delta)\cdot TO(\delta, f^{\sim}{}_{\wp,n})) \cdot(\Sigma\, \kappa^p(\rho)\cdot c(G_{\delta\varepsilon})\cdot O({}^\delta\varepsilon, \varphi^p)) \quad (\delta = \kappa(\rho))$$

(sum over $\kappa \in K(G_\varepsilon/\mathbb{Q})$, $\rho \in \mathscr{E}(G_\varepsilon, \mathbb{Q}_p)$, $\rho \in \mathscr{E}(G_\varepsilon, \mathbb{A}^p{}_f)$)

$$= i(\varepsilon)\cdot|K(G_\varepsilon/\mathbb{Q})|^{-1}\, \Sigma\, \kappa_\infty(\mu - \mu_h) \cdot(\Sigma\, \kappa_p(\rho)\cdot c(G_{\rho\varepsilon})\cdot O({}^\rho\varepsilon, f^{\sim}{}_{\wp,n})) \cdot(\Sigma\, \kappa^p(\rho)\cdot c(G_{\rho\varepsilon})\cdot O({}^\rho\varepsilon, \varphi^p)),$$

κ_∞, κ_p and κ^p are defined by $\mathscr{E}(G_\varepsilon, \mathbb{Q}_v) \to K(G_\varepsilon/\mathbb{Q})^D$, and

$f_{\wp,n}$ is the image of $\tilde{f}_{\wp,n}$ by the base-change homomorphism $\mathcal{H}(G(F^n), K_p(\mathcal{O}_F{}^n)) \rightarrow \mathcal{H}(G(\mathbb{Q}_p), K_p)$ (see 3.2), we have used 3.2 and that for $\rho \in \mathcal{E}(G_\varepsilon, \mathbb{Q}_p)$ is $\beta(h(\rho)) = \beta(b_\varepsilon) \cdot$ the image of ρ by $\mathcal{E}(G_\varepsilon, \mathbb{Q}_v) \rightarrow K(G_\varepsilon/\mathbb{Q})^D$.

1.8 A subscript e will denote "elliptic".

Let $\mathcal{E}$ denote the set of (equivalence classes of) elliptic endoscopic data (H, s, η) for G (K3, thus H is a connected reductive quasi-split group defined over $\mathbb{Q}$, η is an imbedding of the connected L-group ${}^LH^0$ of H into ${}^HG^0$, s belongs to the center Z^H of ${}^LH^0$ and the image of η is the connected component of the centralizer of $\eta(s)$ in ${}^LG^0$).

We have a bijection between the set of (equivalence classes of) pairs ((H, s, η), γ), where $\gamma \in H(\mathbb{Q})_{e,(G,\, H)\text{-reg}}$, and the set of (equivalence classes of) pairs (ε, κ), where $\varepsilon \in G(\mathbb{Q})_e$ and $\kappa \in K(G_\varepsilon/\mathbb{Q})$ (K6, thus γ is "the image" of ε (see below) and since H_γ is an inner form of G_ε, their connected L-groups are isomorphic and so Z^H can be canonically imbedded in Z_ε, and κ is the element of $\pi_0((Z_\varepsilon/Z)^\Gamma)$ containing $s \in Z^H \subset Z_\varepsilon$).

For each (H, s, η) $\in \mathcal{E}$ we choose, once for all, a continuous extension η': ${}^LH^0 \times L_\mathbb{Q} \rightarrow {}^LG^0 \times L_\mathbb{Q}$ of η which commutes with the projections on $L_\mathbb{Q}$ (L is the Langlands group, see 3.10 - η' exists because the center Z of ${}^LG^0$ is connected). Since we have chosen an imbedding $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_v$ for each place, we have a continuous homomorphism $L_{\mathbb{Q}v} \rightarrow L_\mathbb{Q}$ for each place (canonical up to conjugation by an element of $L_\mathbb{Q}$) and η' can be uniquely lifted to a continuous extension η_v': ${}^LH^0 \times L_{\mathbb{Q}v} \rightarrow {}^LG^0 \times L_{\mathbb{Q}v}$ of η which commutes with the projection on $L_{\mathbb{Q}v}$.

We choose local transfer factors $\Delta_v(\gamma_v, \varepsilon_v)$ (v place) (see LS1) and assume that they satisfy the global condition Π_v

$\Delta_v(\gamma_v, \varepsilon_v) = 1$.

Let $\mathscr{E}_\infty$ denote the set of (equivalence classes of) endoscopic data (H, s, η) for G for which ($H_{\mathbb{R}}$, s, η) is elliptic, then $\mathscr{E}_\infty \subset \mathscr{E}$. ε is elliptic at ∞ if and only if (H, s, η) and γ is elliptic at ∞.

1.9 Here we replace n by j (recall that n = jr and $|\omega_{\wp}| = |\omega|^r$) - thus j is divisibel by r.

For each (H, s, η) ∈ $\mathscr{E}_\infty$ we can assume that $\eta(s) \in {}^LT^0$, and we choose, once for all, a Cartan subgroup T_0 of G which is elliptic at infinity, an isomorphism $X_*(T_0) \leftrightarrow X^*({}^LT^0)$ which is such that this, the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X_*(T_0)$ and η(s) determine (H, s, η), and a $h_0 \in X_\infty$ which factorizes through T_0.

For $j \in r\mathbb{N}$ we let $H(\mathbb{Q})^j_\infty$ denote the set of elements in $H(\mathbb{Q})_{s.s.}$ which is the image of some element in $G(\mathbb{Q})^j_\infty$.

For $\gamma \in H(\mathbb{Q})^j_\infty$ we define the sign τ(γ) as follows: choose $\varepsilon \in G(\mathbb{Q})^j_\infty$ such that γ is the image of ε, choose an elliptic Cartan subgroup T of G which contains ε and an isomorphism $X_*(T_0) \leftrightarrow X^*({}^LT^0)$ arising from the relation between γ and ε - that is, $X_*(T_0) \leftrightarrow X^*({}^LT^0)$ comes from the relation between G and ${}^LG^0$, and there is an elliptic Cartan subgroup T^H of H which contains γ, and an isomorphism $X_*(T^H) \leftrightarrow X^*({}^LT^H{}_0)$ which comes from the relation between H and ${}^LH^0$, such that the isomorphism $T^H \leftrightarrow T$ determined by $X_*(T^H) \leftrightarrow X_*({}^LT^{H0}) \leftrightarrow^\eta X^*({}^LT^0) \leftrightarrow X_*(T)$ is defined over $\mathbb{Q}$ and maps γ to φ)) and choose $\mu \in X_*(T)$ such that μ is M_ε-conjugate to a μ satisfying the condition in 1.4, then take $\tau(\gamma) = (\mu - \mu_{h0})(\eta(s))$ (we have identified $X_*(T_0)$, $X_*(T)$ and $X^*({}^LT^0)$). τ(γ) is independent of the choises, and τ(γ) = ±1 because $\eta(s)^2 \in \mathbb{Z}$ since $(H, s, \eta)_{\mathbb{R}}$ is elliptic.

Let $\varepsilon_0 \in T_0(\mathbb{R})$ be such that γ is the $\mathbb{R}$-image of ε_0 (via the isomorphism $X_*(T_0) \leftrightarrow^{\eta} X^*({}^LT^0)$ - ε_0 is determined up to action of the H-Weyl-group). Then

$$\Delta_\infty(\gamma, \varepsilon)\cdot\alpha(\varepsilon)\cdot\kappa_\infty(\mu - \mu_h) = \Delta_\infty(\gamma, \varepsilon_0)\cdot\alpha(\varepsilon_0)\cdot\tau(\gamma)$$

(recall that $\mu_h \in X_*(T)$ where $T \subset G_\varepsilon$ (1.7)).

There is a (finite dimensional) representation ${}^0r_{\wp}$ of ${}^LG^0 \times \mathrm{Gal}(\mathbb{Q}_p^{un}/E_{\wp})$ (unique up to isomorphism) such that it is irreducible on ${}^LG^0$ having extreme ${}^LT^0$-weights Ω_μ and such that $\mathrm{Gal}(\mathbb{Q}_p^{un}/E_{\wp})$ acts trivially on the ${}^LB^0$-highest weight space (K4). By restriction we have a representation ${}^0r_{\wp,j}$ of ${}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/F^j)$.

The function $\tilde{f}_{\wp,j} \in \mathscr{H}(G(F^j), K_p(\mathcal{O}_F{}^j))$ in l.3 is associated to the class function $x \rightarrow |\omega_F{}^j|^{-d/2}$ tr ${}^0r_{\wp,j}(x)$ on ${}^LG^0\times\mathrm{Gal}$ $(\mathbb{Q}_p^{un}/F^j)$ by the Satake transform, $\omega_F{}^j$ is an uniformization element in F^j and $d = 2<\delta, \mu> = \dim S(K)$, here $\mu \in \Omega_\mu$ and δ is the half sum of the positive roots for an order which makes μ dominant (K4).

Let $r_{\wp,j}$ denote the representation of ${}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ obtained by inducing ${}^0r_{\wp,j}$. Then the function $f_{\wp,j} \in \mathscr{H}$ $(G(\mathbb{Q}_p), K_p)$ in l.7 is associated to the class function $x \rightarrow$ $(1/j)\ |\omega^j|^{-d/2}$ tr $r_{\wp,j}(x^j)$ on ${}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ by the Satake transform.

Because $G_{\mathbb{Q}p}$ is unramified and the (H, s, η) that contribute to our sum are such that $H_{\mathbb{Q}p}$ is unramified (see below) we can assume that η_p' is unramified (η_p' differs from a such by an element of $H^1(W_{\mathbb{Q}p}, Z^H)$, this determines a character χ on $H(\mathbb{Q}_p)$, and ${}^Hf_{\wp,j}$ (see below) and Δ_p have to be multiplied by χ), that is, the lifting of a homomorphism η_p: ${}^LH^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \rightarrow {}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$.

Let ${}^0r^H{}_{\wp,j}$ denote the restriction of ${}^0r_{\wp,j}$ to ${}^LH^0\times$ $\mathrm{Gal}(\mathbb{Q}_p^{un}/F^j)$ (via η_p). On the Cartan subgroup ${}^LT^{H0} = \eta^{-1}(T)$ of ${}^LH^0$, ${}^0r^H{}_{\wp,j}$ acts in accordance with Ω_μ (regarded as a $\Omega({}^LG^0$,

$^{L}T^{0}$)-orbit in $X^*(^{L}T^{H}_{0})$. The set

$$\mathscr{H} = \{(\mu - \mu_{h0})(\eta(s)) \mid \mu \in \Omega_\mu\} \subset \{\pm 1\}$$

determines a class decomposition of Ω_μ, and so a decomposition of the representation space of ${}^{0}r^{H}{}_{\wp,j}$, this decomposition respects the action of $^{L}H^{0}\times Gal(\mathbb{Q}_p{}^{un}/F^{j})$ and so we have a decomposition

$${}^{0}r^{H}{}_{\wp,j} = \oplus_{i\in\mathscr{H}}\, {}^{0}r^{\vee H,i}{}_{\wp,j}.$$

Because the restriction $r^{H}{}_{\wp,j}$ of $r_{\wp,j}$ to $^{L}H^{0}\times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$ is obtained by inducing ${}^{0}r^{H}{}_{\wp,j}$ to $^{L}H^{0}\times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$, we have also a decomposition

$$r^{H}{}_{\wp,j} = \oplus_{i\in\mathscr{H}}\, r^{\vee H,i}{}_{\wp,j}.$$

Let $\varphi \in \mathscr{H}(G(\mathbb{A}_f), K)$ be $meas(K/Z_K)^{-1}$ · the characteristic function of K, and $\varphi_p \in \mathscr{H}(G(\mathbb{Q}_p), K_p)$ be meas $(K_p/(Z_K)_p)^{-1}$ · the characteristic function of K_p. Then $\varphi(g_p, g^p) = \varphi_p(g_p)\varphi^p(g^p)$.

For each $(H, s, \eta) \in \mathscr{E}$ that contributes to our sum, $H_{\mathbb{Q}p}$ is unramified, therefore we can choose a hyperspecial subgroup $K^{H}{}_{p}$ of $H(\mathbb{Q}_p)$ such that every $\gamma \in K^{H}{}_{p}$ is the image of some $\varepsilon \in K_p$, and there exists a function φ^H on $H(\mathbb{A}_f)$ such that $\varphi^H \in \mathscr{H}(H(\mathbb{Q}_p), K^{H}{}_{p})$ and such that if $\gamma \in H(\mathbb{A}_f)_{s.s.,(G,H)\text{-reg}}$ then

$$SO_f(\gamma, \varphi^H) = \Delta_f(\gamma, \varepsilon)\, \Sigma\, \kappa_f(\rho)\cdot c(G_{\rho\varepsilon})\cdot O_f({}^{\rho}\varepsilon, \varphi)$$
$$\text{(sum over } \rho \in \mathscr{E}(G_\varepsilon/\mathbb{A}_f))$$
$$\text{if } \gamma \text{ is image of } \varepsilon \in G(\mathbb{A}_f)_{s.s.} \text{ and 0 if not}$$

(see 3.5 and 3.7). Let the function $f^{H}{}_{\wp,j} \in \mathscr{H}(H(\mathbb{Q}_p), K^{H}{}_{p})$ be associated to the class function $x \to (1/j)\, |\omega^{j}|^{-d/2}\, \Sigma_{i\in\mathscr{H}}\, i\ tr\ {}^{\vee}r^{H,i}{}_{\wp,j}$ on $^{L}H^{0}\times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$ by the Satake transform. Then it follows from 3.5 that if $\gamma \in H(\mathbb{Q})^{j}{}_{\infty}$ and $\varepsilon \in G(\mathbb{Q})^{j}{}_{\infty}$ and γ is the image of ε, then

$$SO_p(\gamma, f^H_{\wp,j}*\varphi^H) = \tau(\gamma)\cdot\Delta_p(\gamma, \varepsilon)\ \Sigma\ \kappa_p(\rho)\cdot c(G_{\rho\varepsilon})\cdot O_p({}^\rho\varepsilon, f_{\wp,j})$$
$$\text{(sum over } \rho \in \mathscr{E}(G_\varepsilon/\mathbb{Q}_p)).$$

Furthermore it follows from 3.4 that there exists a function f^H_ξ on $H(\mathbb{R})$ such that

$$SO_\infty(\gamma, f^H_\xi) = \Delta_\infty(\gamma, \varepsilon)\cdot\alpha(\varepsilon_0)$$
$$\text{for } \gamma \in H(\mathbb{R})_e \text{ and } 0 \text{ for } \gamma \in H(\mathbb{R})_{s.s.}\backslash H(\mathbb{R})_e.$$

In the above $SO_\nu(\gamma, f)$ (ν place) denotes the stable orbital integral at $\gamma \in H(\mathbb{Q}_\nu)$ of the function f on $H(\mathbb{Q}_\nu)$, that is,

$$SO_\nu(\gamma, f) = \Sigma\ c(H_{\rho\gamma})\cdot O_\nu({}^\rho\gamma, f) \text{ (sum over } \rho \in \mathscr{E}(H_\gamma/\mathbb{Q}_\nu)).$$

The number of stable conjugacy classes of elements $\gamma \in H(\mathbb{Q})_{s.s.,(G, H)\text{-reg}}$ which are the image of a given $\varepsilon \in G(\mathbb{Q})_{s.s.}$ is $\lambda(H, s, \eta) = |Aut(H, s, \eta)/H_{ad}(\mathbb{Q})|$ (K6). If we denote the number $\lambda(H, s, \eta)^{-1}\cdot\tau(G)\cdot\tau(H)^{-1}$ by $\iota(G, H)$ (see K3), then it follows from 3.3 that

$$\lambda(H, s, \eta)^{-1}\cdot i(\varepsilon)\cdot|\kappa(G_\varepsilon/\mathbb{Q})|^{-1}\cdot\tau(G_\varepsilon)_K$$
$$= \iota(G, H)\cdot i(\gamma)\cdot|\kappa(H_\varepsilon/\mathbb{Q})|^{-1}\cdot\tau(H_\gamma)_K.$$

1.10 It follows from 3.4 and 3.5 that we can extend the summation from $\mathscr{E}_\infty$ to $\mathscr{E}$ and from $H(\mathbb{Q})^j_\infty$ to $H(\mathbb{Q})_e$.

Let $\kappa(H, \eta')$ be the character of $Z(\mathbb{A})$ constructed in LS1 (p. 252 - in this paper however only on the identity component of Z). It is determined by η' and satisfies $\Delta_\nu(z\gamma, z\varepsilon) = \kappa(H, \eta')_\nu(z)\cdot\Delta_\nu(\gamma, \varepsilon)$ (ν place). Let ν^H_∞ be the character $\nu\cdot\kappa(H, \eta')_\infty$ of $Z(\mathbb{R})$ and let ι^H_f be the character $\kappa(H, \eta')|Z_K$ of Z_K.

We let $F^H_{\wp,j}$ denote the function on $H(\mathbb{A})$ defined by $F^H_{\wp,j}(h) = f^H_\xi(h_\infty)\cdot r(F^H_{\wp,j}*\varphi^H_p)(h_p)\cdot\varphi^{Hp}(h^p)$. For $z \in Z(\mathbb{R})$ we have $F^H_{\wp,j}(zh) = \nu^H_\infty(z)\cdot F^H_{\wp,j}(h)$ and for $z \in Z_K$ we have $F^H_{\wp,j}(zh) = \iota^H_f(z)\cdot F^H_{\wp,j}(h)$.

Σ ... (sum over $\gamma \in H(\mathbb{Q})_e/\sim_K$) is the stable elliptic part of the trace of $F^H_{\wp,j}$ (see L8 or K6).

Let $\Phi(H)_e$ denote the set of (equivalence classes of) elliptic (essentially) tempered admissible homomorphisms ψ: $L_\mathbb{Q} \to {}^LH^0 \times L_\mathbb{Q}$ such that $\chi_{\psi\infty}|Z(\mathbb{R}) = (\nu^H{}_\infty)^{-1}$ and $\chi_{\psi f}|Z_K = (\iota^H{}_f)^{-1}$ ($\chi_{\psi v}$ is the character of $Z^H(\mathbb{Q}_v)$ accociated to $\psi_v \in \Phi(H_v)$), then the stable tempered cuspital part of the trace is

$$\Sigma\, d_\psi^{-1}\, \Sigma\, n_\pi \operatorname{tr} \pi(F^H_{\wp,j})\ (\psi \in \Phi(H)_e, \pi \in \Pi(\psi)),$$

here d_ψ is the number of (global) equivalence classes in the local equivalence class of ψ (d_ψ different classes of $\Phi(H)_e$ parametrize $\Pi(\psi)$), n_π is the stable multiplicity of π and $\pi(f)$ is the operator $\int \pi(h)f(h)\, dh$ (integral over $Z(\mathbb{R})\backslash H(\mathbb{A})/Z_K$). This part of the stable trace is "contained" in the stable elliptic part of the trace (for all this, see 3.10).

1.11 Because $\varphi^H{}_p \in \mathscr{H}(H(\mathbb{Q}_p), K^H{}_p)$ and is non-zero, $\operatorname{tr} \pi_p(\varphi^H{}_p) \neq 0 \Rightarrow \pi_p$ has a non-zero vector fixed by $K^H{}_p$. Hence ψ_p is unramified, and in this case exactly one π_p in $\Phi(\psi_p)$ has a non-zero vector fixed by $K^H{}_p$.

It follows from 3.8 that we can restrict the summation to those ψ for which $\varphi = \eta'\circ\psi$ is elliptic and admissible for G (φ is elliptic because φ_∞ is elliptic), this set is denoted by $\Psi(H)_{G\text{-}e}$.

1.12 Let $\Phi(G)_e$ denote the set of (equivalence classes of) elliptic tempered admissible homomorphisms φ: $L_\mathbb{Q} \to {}^LG^0 \times L_\mathbb{Q}$ such that $\chi_{\varphi\infty} = \nu^{-1}$ and $\chi_{\varphi f}|Z_K = 1$.

Let $\psi \in \Phi(H)_{G\text{-}e}$, then $\varphi = \eta'\circ\psi \in \Phi(G)_e$, we let Π^H and Π denote $\Pi(\psi)$ and $\Pi(\varphi)$. We can assume that ψ_∞ and φ_∞ are elliptic (see 3.8) and, by replacing ψ_∞ by an equivalent, we can assume that $\psi_\infty(\mathbb{C}^\times) \subset {}^LT^{H0} \times \mathbb{C}^\times$ and $\psi_\infty(\tau) =$

$h\times\tau$ for som $h \in Nm_{LH0}({}^LT^{H0})$, then $\varphi_\infty(\mathbb{C}^\times) \subset {}^LT^0\times\mathbb{C}^\times$ and $\varphi_\infty(\tau) = g\times\tau$ for some $g \in Nm_{LG0}({}^LT^0)$. $\varphi_\infty(\tau)$ determines an action ι' on ${}^LT^0$ and for this action of the non-trivial elements in $Gal(\mathbb{C}/\mathbb{R})$, ${}^LT^0\times Gal(\mathbb{C}/\mathbb{R})$ is the L-group of the fundamental Cartan subgroup $(T_0)_\mathbb{R}$ of $G_\mathbb{R}$ (via the isomorphism $X_*(T_0) \leftrightarrow X^*({}^LT^0)$). To φ_∞ is associated a $\Omega(G(\mathbb{C}), T_0(\mathbb{C}))$-orbit Ω_λ of continuous regular charactors of $T_0(\mathbb{R})$ (Bo - note that the action of the elements of $\Omega(G(\mathbb{C}), T_0(\mathbb{C}))$ on $T_0(\mathbb{C})$ is defined over $\mathbb{R}$ because $(T_0\cap G_{der})(\mathbb{R})$ is compact) and so a set of discrete series representations of $G(\mathbb{R})$, this set is just $\Pi(\varphi_\infty) = \Pi_\infty$.

$\varphi_\infty|\mathbb{C}^\times$ has the form $z \rightarrow z^{\Lambda 0}\overline{z}^{\iota'\Lambda 0}\times z$, where $\Lambda_0 \in X_*({}^LT_0)\otimes\mathbb{R}$ (in fact $\Lambda_0 \in \frac{1}{2}X_*({}^LT_0)$ because $\Lambda_0 \in \delta+X_*({}^LT_0)$ and $\Lambda_0|Z$ = the rational character $\nu^{-1} - \delta$ is the half sum of the positive roots of G w.r.t. T_0 for some order. Since Λ_0 is non-singular it lies in an open Weyl chamber, let μ_0 be the weight in Ω_μ ($\subset X_*({}^LT_0)$) lying in the closure of the opposite chamber. The $\Omega({}^LH^0, X^*({}^LT^{H0}))$-orbit of μ_0 (regarded as a weight in $X^*({}^LT^{H0})$) is determined by (the equivalence class of) ψ_∞.

Because $(H_\mathbb{R}, s, \eta)$ is elliptic, $\eta(s)^2 \in Z$ and $(H_\mathbb{R}, s, \eta)$ can be constructed from $(T_0)_\mathbb{R}$ and the character κ_∞ of $\mathcal{E}(T_0/\mathbb{R}) = X^*({}^LT^0{}_{der})/2X^*({}^LT^0{}_{der})$ (the Tate-Nakayama isomorphism, note that ι' acts on $X^*({}^LT^0{}_{der})$ by $\mu \rightarrow -\mu$) given by $\kappa_\infty(\{\mu\}) = \mu(\eta(s))$ $(= \pm 1)$. The restriction of κ_∞ to $\mathcal{D}(T_0/\mathbb{R}) = \Omega(G(\mathbb{C}), T_0(\mathbb{C}))/\Omega(G(\mathbb{R}), T_0(\mathbb{R}))$ has image $\mathscr{H}$ (because $\kappa_\infty(\{\omega\}) = (\omega\mu_{h0} - \mu_{h0})(\eta(s)))$.

(H, s, η) and (the equialence class of) ψ_∞ determines a class decomposition of Π_∞:

$$\Pi_\infty = \cup_{i\in\mathscr{H}}\Pi^i{}_\infty,$$

where $\Pi^i{}_\infty = \{\pi \mid \exists\omega \in \Omega(G(\mathbb{C}), T_0(\mathbb{C})): \kappa_\infty(\{\omega\}) = i$ and π is attached to $\lambda_0\circ\omega\}$, here λ_0 is the character of $T_0(\mathbb{R})$ determined by Λ_0.

We choose a function f^G_ξ: $G(\mathbb{R}) \to \mathbb{C}$ such that $SO_\infty(\varepsilon, f^G_\xi) = \alpha(\varepsilon)$ (3.6), and let

$$m(\Pi^H_\infty) = \Sigma <1, \pi> \mathrm{tr}\ \pi(f^H_\xi) \text{ (sum over } \pi \in \Pi^H_\infty)$$

and

$$m(\Pi_\infty) = \Sigma <1, \pi> \mathrm{tr}\ \pi(f^G_\xi) \text{ (sum over } \pi \in \Pi_\infty).$$

Since we can assume that $m(\Pi^H_\infty) \neq 0$, we can (3.8) assume that ψ_∞ and φ_∞ are elliptic, and it follows from 3.6 that

$$m(\Pi^H_\infty)\cdot i = e_\infty \cdot m(\Pi_\infty) <\eta(s), \Pi^i_\infty>$$

for $i \in \mathscr{H}$.

If we in the deccomposition $r^H_{\wp,j} = \oplus_{i\in\mathscr{H}} {}^\vee r^{H,i}_{\wp,j}$, instead of letting the summand indexed by $1 \in \mathscr{H}$ be that containing μ_{h0}, now be that containing μ_0, we get a new decomposition:

$$r^H_{\wp,j} = \oplus_{i\in\mathscr{H}} r^{H,i}_{\wp,j},$$

and

$$r^{H,i}_{\wp,j} = {}^\vee r^{H,i\eta}_{\wp,j},$$

where $\eta = (\mu_0 - \mu_{h0})(\eta(s)))$ $(= \pm 1)$.

Now we have

$$m(\Pi^H_\infty)\ \Sigma_{j=1}^{\infty}{}_{,r|j}\ |\omega|^{js}/j\ \mathrm{tr}\ \pi_p(r\cdot f^H_{\wp,j})$$
$$= m(\Pi^H_\infty)\ \Sigma_{i\in\kappa}\, i \log L(s - d/2, \pi_p, {}^\vee r^{H,i}_{\wp,r})$$
$$= m(\Pi^H_\infty)\ \Sigma_{i\in\kappa}\, i\cdot\eta \log L(s - d/2, \pi_p, r^{H,i}_{\wp,r})$$
$$= e_\infty\cdot m(\Pi^H_\infty)\ \Sigma_{i\in\kappa} <\eta(s), \Pi^{i\eta}_\infty> \log L(s - d/2, \pi_p, r^{H,i}_{\wp,r}),$$

here we have used that $\mathrm{tr}\ {}^\vee r^{H,i}_{\wp,r}(\psi_p(\sigma)^j) = 0$ for j not divsibel by r (because ${}^\vee r$ is induced from a subgroup of index r). We recall that the L-function associated to an unramified admissible homomorphism φ: $W_{\mathbb{Q}_p} \to {}^LG^0\times W_{\mathbb{Q}_p}$ (that is, an admissible homomorphism $\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \to {}^LG^0\times \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ or a semisimple ${}^LG^0$-conjugacy class in ${}^LG^0\times \sigma$) and a (finite dimensional) representation r of ${}^LG^0\times$Gal

($\mathbb{Q}_p^{un}/\mathbb{Q}_p$) is defined by

$$L(s, \Pi(\varphi), r) = L(s, \pi, r) = \det(1 - |\omega|^s\, r(\varphi(\sigma)))^{-1}$$

or

$$\log L(s, \pi(\varphi), r) = \Sigma_{j=1}^{\infty} |\omega|^{js}/j \operatorname{tr} r(\varphi(\sigma)^j)$$

- π is the representation in $\Pi(\varphi)$ having a non-zero vector fixed by the maximal compact subgroup K_p and σ is the Frobenius in $\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$.

Since the class $\overline{M}_{\overline{\mathbb{Q}}} \in \mathscr{M}(\overline{\mathbb{Q}})$ is left fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ (recall that E by definition is the smallest Galois extension of $\mathbb{Q}$ having this property), we can construct a (finite dimensional) representation ${}^0r$ of ${}^LG^0 \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ (unique up to isomorphism) such that it is irreducible on ${}^LG^0$ having extreme ${}^LT^0$-weights Ω_μ and such that $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts trivially on the ${}^LB^0$-highest weight space (the construction is analogous to the earlier construction of ${}^0r_{\mathscr{p},r}$ associated to the class $\overline{M}_{\mathscr{p}} \in \mathscr{M}(\overline{\mathbb{Q}}_p)$ left fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_{\mathscr{p}})$, recall that Ω_μ is the Weyl-group orbit in $X^*({}^LT^0)$ associated to $\overline{M}_{\overline{\mathbb{Q}}}$ and that E is the smallest Galois extension of $\mathbb{Q}$ having the property that if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ and $\mu \in \Omega_\mu$ then $\sigma\mu \in \Omega_\mu$). By induction we have a representation r of ${}^LG^0 \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$, and by lift we have a representation, also denoted r, of ${}^LG^0 \times L_{\mathbb{Q}}$.

The restriction r^H of r to ${}^LH^0 \times L_{\mathbb{Q}}$ (via η') has a decomposition formed in an analogous way as before

$$r^H = \oplus_{i \in \mathscr{H}} r^{H,i}$$

(the summand indexed by $1 \in \mathscr{H}$ contains μ_0 and the decomposition is determined by the equivalence class of ψ_∞).

The restriction of the representation $r^{H,i}$ to ${}^LH^0 \times L_{\mathbb{Q}}$ is the lifting of the representation $\oplus_{\overline{\mathscr{p}}|p}\, r^{H,i}{}_{\overline{\mathscr{p}},r}$ to ${}^LH^0 \times \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ (note that we shall use different imbeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ for the construction of the various $r^{H,i}{}_{\overline{\mathscr{p}},r}$, $\overline{\mathscr{p}}|p$). The construction can also be carried out in the following way: choose $\tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ such that its restriction to E transforms $\mathscr{p}$ to $\overline{\mathscr{p}}$, since τ normalizes $Gal(\overline{\mathbb{Q}}/E)$, $1\times\tau \in {}^LG^0\times Gal(\overline{\mathbb{Q}}/E)$ normalizes ${}^LG^0\times Gal(\overline{\mathbb{Q}}/E)$, and if we restrict ${}^0r\circ ad(1\times\tau)$ to ${}^LG^0 \times W_{E\mathscr{p}}$ (via $W_{E\mathscr{p}} \rightarrow W_E \rightarrow Gal(\overline{\mathbb{Q}}/E)$), it will be the lifting of a representation of ${}^LG^0\times Gal(\mathbb{Q}_p^{un}/E_{\mathscr{p}})$, if we induce this to ${}^LG^0\times Gal(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ and then restrict to ${}^LH^0\times Gal(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$, we get $\oplus_{i\in\mathscr{H}} r^{H,i}{}_{\overline{\mathscr{p}},r}$ (recall that r is independent of $\mathscr{p}|p$ since E is Galois). We therefore have

$$\Pi_{\overline{\mathscr{p}}|p} L(s - d/2, \pi_p, r^{H,i}{}_{\overline{\mathscr{p}},r}) = L(s - d/2, \pi_p, r^{H,i}).$$

If follows from 3.7 that for $\psi \in \Phi(H)_{G\text{-}e}$ is

$$\Sigma <1, \pi>_f tr\, \pi(\varphi^H) = e_f \Sigma <\eta(s), \pi>_f tr\, \pi(\varphi)$$

(sum over resp. $\pi \in \Pi(\psi)_f$ and $\pi \in \Pi(\varphi)_f$), the pairing $< , >_f = \zeta_\varphi\times\Pi(\varphi)_f \rightarrow \mathbb{C}$, where $\zeta_\varphi = S_\varphi/(S_\varphi)^0Z$ (= S_φ/Z because $(S_\varphi)^0 \subset Z$ for φ elliptic) and where $S_\varphi = \{g \in {}^LG^0 \mid ad(g)\circ\varphi$ differs from φ by a continuous locally trivial 1-cocycle of $L_{\mathbb{Q}}$ in $Z\}$ is defined via $\zeta_\varphi \rightarrow \zeta_{\varphi v}$ and $\Pi(\varphi) \rightarrow \Pi(\varphi_v)$ by letting $<s, \pi>_f = \Pi_{v\neq\infty} <s_v, \pi_v>_v$ ($< , >_f \cdot < , >_\infty$ is canonical). $e_\infty\cdot e_f = 1$ by 3.9.

1.13 Let ${}_G\Phi(H)_{G\text{-}e}$ denote the set $\Phi(H)_{G\text{-}e}$, where the equivalence relation is replaced by Z-equivalence, and for $\psi \in {}_G\Phi(H)_{G\text{-}e}$ let ${}_G\zeta_\psi = {}_GS_\psi/({}_GS_\psi)^0Z$, where ${}_GS_\psi$ is obtained if we in the definition of S_ψ replace Z^H by Z.

Since $\iota(G, H) = (H, s, \eta)^{-1}\cdot\tau(G)\cdot\tau(H)^{-1}$ and the number of Z-equivalence classes in the equivalence class of $\Phi(H)_{G\text{-}e}$ containing ψ is $|{}_G\zeta_\psi|\cdot|\zeta_\psi|^{-1}\cdot\tau(G)\cdot\tau(H)^{-1}$ (K3) we have

$$\Sigma\, \iota(G, H)\, \Sigma\, |\zeta_\psi|^{-1} (...) = \Sigma\, \lambda(H, s, \eta)^{-1}\, \Sigma\, |{}_G\zeta_\psi|^{-1} (...)$$

(sum over (H, s, η) ∈ $\mathscr{E}$ and $\psi \in \Phi(H)_{G\text{-}e}$ resp. ${}_G\Phi(H)_{G\text{-}e}$).

Let ~ denote the conjugation on ζ_φ. If ψ and ψ' are Z-equivalent, then η'∘ψ and η'∘ψ' are equivalent. We can therefore for $\varphi \in \Phi(G)_e$ and $\{\bar{s}Z\} \in \zeta_\varphi/\sim$ restrict the above sum to those (H, s, η) ∈ $\mathscr{E}$ and $\psi \in {}_G\Phi(H)_{G\text{-}e}$ such that if φ' = η'∘ψ, then φ' ~ φ and the canonical bijection $\zeta_{\varphi'}/\sim \leftrightarrow \zeta_\varphi/\sim$ maps {η(s)Z} to $\{\bar{s}Z\}$, and then summarize over φ and $\{\bar{s}Z\}$.

1.14 For given $\varphi \in \Phi(G)_e$ and $\{\bar{s}Z\} \in \zeta_\varphi/\sim$ there exists ((H, s, η), ψ) which maps to (φ, $\{\bar{s}Z\}$) and the second line of (13) is independent of ((H, s, η), ψ). This follows from the following way to define the first parenthesis in (14).

By replacing φ by an equivalent we can assume that $\bar{s} \in {}^LT^0$ and $\varphi_\infty(\mathbb{C}^\times) \subset {}^LT^0 \times \mathbb{C}^\times$. We construct (H, s, η) ∈ $\mathscr{E}$ and ψ ∈ Φ(H) such that φ' = η'∘ψ, and $\bar{s}Z$ = η(s)Z: take ${}^LH^0 = (\text{centralizer}_{LG0}(\bar{s}))^0$ and the action of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on ${}^LH^0$ to be given by $k_\sigma g_w \times \sigma$, where w → σ and $\varphi(w) = g_w \times \sigma$ ($g_w \times \sigma \in \mathrm{Nm}_{LG0\times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}({}^LH^0)$) and $k_\sigma \in {}^LH^0$ chosen such that the action on ${}^LH^0$ leaves ${}^LT^{H0} = {}^LT^0$ and ${}^LB^{H0} = {}^LH^0 \cap {}^LB^0$ invariant and permutes the root vectors (used in the construction of ${}^LG^0 \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) associated to the ${}^LB^{H0}$-simple roots of ${}^LT^{H0}$ in ${}^LH^0$ (the action is uniquely determined by these requirements), then $s = \bar{s}$, η = the inclusion ${}^LH^0 \subset {}^LG^0$ and $\psi(w) = \varphi(w)\eta'(w)^{-1} \times w$. $\psi_\infty|\mathbb{C}^\times$ determines a $\mu_0 \in X^*({}^LT^{H0})$ (see 1.12) and so a decomposition $r = \oplus_{i\in\mathscr{H}} r^{H,i}$. If we had chosen another pair (φ', $\bar{s}'$) equivalent to (φ, $\bar{s}$), there would exist a $g \in {}^LG^0$ such that φ'(w) = c(w)·(ad(g)∘φ)(w) (c a continuous locally trivial 1-cocycle of $L_\mathbb{Q}$ in Z) and $\bar{s}'Z = \mathrm{ad}(\bar{s})Z$. If we define the map β(g): ${}^LH^0 \times \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \to {}^LH'^0 \times \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ by $h \to ghg^{-1}$ and $\sigma \to (t')^{-1} \times \sigma$, where $t' \in Z^{H'}$ is such that $\eta'_p(\sigma) = t' g \eta_p(\sigma) g^{-1}$, we have tr $r^{H'} \circ \beta(g)$ = tr r^{H} and $\psi'_p = z(\beta(g)\circ\psi_p)z^{-1}$ (if $c(\sigma) = z\sigma(z)^{-1}$, $z \in Z$), furthermore since ${}^{L}T^{H'0} = \beta(g)({}^{L}H^{H0})$, $\psi'_\infty | \mathbb{C}^\times = \beta(g)\circ\psi_\infty|\mathbb{C}^\times$ and $\mu'_0 = (g)(\mu_0)$, we have $\mathscr{H}' = \mathscr{H}$ and tr $r^{H',i}\circ\beta(g)$ = tr $r^{H,i}$ and so $L(s - d/2, \pi_p, r^{H',i}) = L(s - d/2, \pi_p, r^{H,i})$. We shall also use that $< , \pi>: \zeta_{\varphi\nu} \to \mathbb{C}$ (for $\pi \in \Pi(\varphi_\nu)$, ν place) is a class function.

Since

$$\Sigma\ (H, s, \eta)^{-1}\ \Sigma\ |{}_G\zeta_\psi|^{-1} = |(\zeta_\varphi)_{\bar{s}Z}|^{-1}$$

$$\text{(sum over } (H, s, \eta) \in \mathscr{E},\ \psi \in {}_G\Phi(H)_{G\text{-}e}),$$

where the summation is taken over the above ((H, s, η), ψ) (K3) and since

$$\Sigma\ |\zeta_\varphi|^{-1} = |(\zeta_\varphi)_{\bar{s}Z}|^{-1} \text{ (sum over } sZ \in \zeta_\varphi,\ sZ \sim \bar{s}Z)$$

we get (14).

2 The formal part of the proof

$$\Sigma_{\wp|p} \log Z(s, S_{\wp}(K), \xi)$$

(1) $= \Sigma_{\wp|p} \Sigma_{j=1}^{\infty} |\omega_{\wp}|^{js}/j\ \Sigma\ tr(\Phi_{\wp}^{j})_x$ (sum over $x \in S_{\wp}(K)(\kappa^j)$)

(2) $=$ ---"--- $\Sigma_{\varphi} \Sigma\ tr\ \xi(\varepsilon)\ |(I_{\varphi})_{\varepsilon}\backslash(Y^j{}_p \times Y^p)|$ (sum over $\varepsilon \in I_{\varphi}/\sim_K$)

(3) $=$ ---"--- $\Sigma\ tr\ \xi(\varepsilon)\ meas((I_{\varphi})_{\varepsilon} Z_K \backslash (G^{\sigma}{}_{\delta}(\mathbb{Q}_p) \times G_{\gamma}(\mathbb{A}^p{}_f)))$
$\cdot TO(\delta, f\tilde{}_{\wp,n}) \cdot O(\gamma, \varphi^p)$
(sum over $\{(\varphi, \varepsilon)\}$ j-perm. K-equ. cl.)

(4) $=$ ---"--- $\Sigma\ tr\ \xi(\varepsilon)\ \Sigma$ ---"---
(sum over $\varepsilon \in G(\mathbb{Q})^n{}_{\infty}/\sim_K$, $\{(\varphi, \varepsilon)\}$ j-perm. K-equ. cl., $\varepsilon' \sim_K \varepsilon$)

(5) $=$ ---"--- $\Sigma_{\varepsilon \in \{fav.rep.\}}\ c_{\infty}\ tr\ \xi(\varepsilon)\ \Sigma\ meas((G_{\varepsilon})_{\varphi}(\mathbb{Q}) Z_K \backslash$
$(G_{\varepsilon})_{\varphi}(\mathbb{A}_f)) \cdot c_p \cdot TO(\delta, f\tilde{}_{\wp,n}) \cdot c^p \cdot O(\gamma, \varphi^p)$
(sum over $\varphi \in P_{\varepsilon}$)

(6) $=$ ---"--- $\Sigma_{\varepsilon \in \{fav.rep.\}}\ \alpha(\varepsilon) \cdot \tau(G_{\varepsilon})_K\ \Sigma\ c_p \cdot TO(\delta, f\tilde{}_{\wp\ n})$
$\cdot c^p \cdot O(\gamma, \varphi^p)$
(sum over $\varphi \in P_{\varepsilon}$)

(7) $=$ ---"--- $\Sigma_{\varepsilon \in \{fav.rep.\}}\ \alpha(\varepsilon) \cdot \tau(G_{\varepsilon})_K \cdot i(\varepsilon) \cdot |K(G_{\varepsilon}/\mathbb{Q})|^{-1}$
$\cdot \Sigma\ \kappa_{\infty}(\mu - \mu_h))$
$\cdot (\Sigma\ \kappa_p(\rho) \cdot c(G_{\rho\varepsilon}) \cdot O({}^{\rho}\varepsilon, f\tilde{}_{\wp,n}))$
$\cdot (\Sigma\ \kappa^p(\rho) \cdot c(G_{\rho\varepsilon}) \cdot O({}^{\rho}\varepsilon, \varphi^p)))$
(sum over $\kappa \in K(G_{\varepsilon}/\mathbb{Q})$, $\rho \in \mathscr{E}(G_{\varepsilon}/\mathbb{Q}_p)$, $\rho \in \mathscr{E}(G_{\varepsilon}/\mathbb{A}^p{}_f)$)

$$(8) = \text{----"----}\ (1/r)\ \Sigma\ \Sigma\ i(\varepsilon)\cdot|K(G_\varepsilon/\mathbb{Q})|^{-1}\cdot\tau(G_\varepsilon)_K$$
$$\cdot(\Delta_\infty(\gamma, \varepsilon)\cdot\alpha(\varepsilon)\cdot(\Sigma\ \kappa_\infty(\mu - \mu_h))$$
$$\cdot(\Delta_p(\gamma, \varepsilon)\cdot r\ \Sigma\ ...)$$
$$\cdot(\Delta^p(\gamma, \varepsilon)\ \ \ \Sigma\ ...)$$
(sum over $\varepsilon \in (G(\mathbb{Q})^n{}_\infty/\sim_K, ...)$

$$(9) = \Sigma_{\wp|p}\ \Sigma_{j=1}^{\infty}{}_{r\,|j}\ |\omega_\wp|^{js}/j\ \Sigma\ \iota(G, H)\ \Sigma\ i(\gamma)\cdot|K(H_\gamma/\mathbb{Q})|^{-1}$$
$$\cdot\tau(H_\gamma)_K\cdot SO_\infty(\gamma, f^H{}_\xi)$$
$$\cdot SO_p(\gamma, f^H{}_{\wp,j} * \varphi^H{}_p)\cdot SO^p(\gamma, \varphi^{Hp})$$
(sum over $(H, s, \eta) \in \mathscr{E}_\infty$, $\gamma \in H(\mathbb{Q})^i{}_\infty/_{\sim K}$)

$$(10) = \text{---"---}\ \Sigma\ \iota(G, H)\ \Sigma\ |\zeta_\psi|^{-1}\ \Sigma <1, \pi>\ \mathrm{tr}\ \pi(F^H{}_{\wp,j})$$
+ non-temp.-cusp. part
(sum over $(H, s, \eta) \in \mathscr{E}$, $\psi \in \Phi(H)_e$, $\pi \in \Pi(\psi)$)

$$(11) = \text{non-temp.-cusp. part} + \Sigma\ \iota(G, H)\ \Sigma\ |\zeta_\psi|^{-1}$$
$$\cdot(\Sigma <1, \pi>\ \mathrm{tr}\ \pi(f^H{}_\xi))$$
$$\cdot(\Sigma_{\wp|p}\ \Sigma_{j=1}^{\infty}\ |\omega_\wp|^{js}/j\ \mathrm{tr}\ \pi_p(r\cdot f^H{}_{\wp,j}))$$
$$\cdot(\Sigma <1, \pi>\ \mathrm{tr}\ \pi(\varphi^H))$$
(sum over $(H, s, \eta) \in \mathscr{E}$, $\psi \in \Phi(H)_{G\text{-}e}$, $\pi \in \Pi^H{}_\infty$, $\wp|p$, $\pi \in \Pi^H{}_f$)

$$(12) = \text{non-temp.-cusp. part} + \Sigma\ \iota(G, H)\ \Sigma\ |\zeta_\psi|^{-1}\ m(\Pi_\infty)$$
$$\cdot(\Sigma <\eta(s), \Pi^{i\eta}{}_\infty>\ \log L(s - d/2, \pi_p, r^{H,i}))$$
$$\cdot(\Sigma <\eta(s), \pi>\ \mathrm{tr}\ \pi(\varphi))$$
(sum over $(H, s, \eta) \in \mathscr{E}$, $\psi \in \Phi(H)_{G\text{-}e}$, $i \in \mathscr{H}$, $\pi \in \Pi_f$)

$$(13) = \text{non-temp.-cusp. part} + \Sigma\ \Sigma\ \Sigma\ \lambda(H, s, \eta)^{-1}\ \Sigma\ |{}_G\zeta_\psi|^{-1}$$
$$\cdot m(\Pi_\infty)\cdot\text{last two lines of (12)}$$
(sum over $\varphi \in (G)_e$, $\{\bar{s}Z\} \in \zeta_\varphi/\sim$, $(H, s, \eta) \in \mathscr{E}$, $\psi \in \Phi_G(H)_{G\text{-}e}$,
$((H, s, \eta), \psi) \to (\varphi, \{\bar{s}Z\})$)

$$(14) = \text{non-temp.-cusp. part} + \log \Pi_{\varphi\in\Phi(G)e} \Pi_{sZ\in\zeta\varphi} (\Pi_{i\in\kappa} L(s - d/2, \pi_p, r^{H,i})^b)^a)$$

$b = \langle s, \Pi^{i\eta}_{\infty}\rangle$ and $a = |\zeta_\varphi|^{-1}\, m(\Pi_\infty)\cdot(\Sigma_{\pi\in\Pi f}\langle s, \pi\rangle \operatorname{tr} \pi(\varphi))$

3 List of conjectures

3.1 If E is unramified over p, G is quasi-split over $\mathbb{Q}_p$, K_p is hyperspecial and if K^p is so small that S(K) has good reduction modulo the prime ideal $\wp$ of E over p (that is, the reduced variety $S_\wp(K)$ exists and is proper and smooth), then the set of equivalence classes of permissible homomorphisms φ: $\mathscr{L} \rightarrow G$ can be put into a bijective correspondance with a class decomposition of $S_\wp(K)(\bar{\kappa})$ in which each class is invariant under the Frobenius action, and the class corresponding to φ can be put into a bijective correspondance with $X_\varphi(K)$ such that the action of the Frobenius on the class corresponds to the action of Φ on $X_\varphi(K)$.

The proof of this conjecture seems to be the most difficult part of the theory, and I will sketch the proof in some of the cases in which the Shimura variety S(K) parametrizes a family of polarized abelian varieties with endomorphism- and level structure (of type K). G is the group of symplectic similitudes on a $\mathbb{Q}$-vector space V w.r.t. a non-degenerate alternating bilinear form ψ (on V) and the action (on V) of a simple $\mathbb{Q}$-algebra D of degree d^2 over its center L, that is, $G = \{g \in GL_D(V) \mid \psi(gu,gv) = \psi(c(g)u, v)\ c(g) \in L_0)$ - D is endowed with a positive involution *, ψ satisfies $\psi(xu, v) = \psi(u, x^*v)$ $(x \in D)$ and L_0 is the fixed field of * on L. There exist a homomorphism h: $\underline{S} \rightarrow G_\mathbb{R}$ defined over $\mathbb{R}$ such that the corresponding Hodge structure on $V \otimes \mathbb{R}$ is of type (1, 0)+(0, 1) and such that $\psi(u, h(i)v)$ is symmetric and positive definite. We choose an order $\mathcal{O}_D$ of D and an $\mathcal{O}_D$-invariant lattice $V_\mathbb{Z}$ of V, and we choose p such that p is unramified in D, $D \otimes \mathbb{Q}_p$ is a product of matrix algebras, $\mathcal{O}_D \otimes \mathbb{Z}_p$ is a maximal order and

ψ: $V_{\mathbb{Z}p} \times V_{\mathbb{Z}p} \rightarrow \mathbb{Z}_p$ is perfect, then $*(\mathcal{O}_D \otimes \mathbb{Z}_p) = \mathcal{O}_D \otimes \mathbb{Z}_p$ and we take $K_p = G(\mathbb{Q}_p) \cap \mathrm{End}_{\mathcal{O}D}(V_{\mathbb{Z}p})$. If K^p is sufficiently small then the pair (G, h) and $K = K_p \cdot K^p$ defines a Shimura variety S(K) satisfying all our wanted properties. The definition field E of S(K) is the subfield of $\overline{\mathbb{Q}}$ generated by the image of the linear map t: $D \rightarrow \overline{\mathbb{Q}}$ given by t(x) = $tr(x|V^{1,0}{}_h)$.

S(K)(E) can be put into a bijective correspondance with the set of (isomorphy classes of) quadruples $(A, \iota, \Lambda, \overline{\eta})$, where A is an abelian variety over $\mathbb{C}$ up to isogeny, ι is a homomorphism $D \rightarrow \mathrm{End}(A)$ such that tr(x|Lie*A) = t(x) for $x \in D$ (Lie*A is the cotangent space of A), Λ is a L_0-homogeneous polarization on A which induces the involution * on D and $\overline{\eta}$ is an equivalence class for the action of K of $D \otimes \mathbb{A}_f$-module isomorphisms η: $H^1(A, \mathbb{A}_f) \rightarrow^{\sim} V \otimes \mathbb{A}_f$ which transform ψ to the form on $H^1(A, \mathbb{A}_f)$ induced by a polarization in Λ up to multiplication by an element of $L_0 \otimes \mathbb{A}_f$.

$S_{\wp}(K)(\overline{\kappa})$ can be put into a bijective correspondance with the set of (isomorphy classes of) quadruples $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \overline{\eta}^{\sim})$, where $A^{\sim}$ is an abelian variety over $\overline{\kappa}$ up to isogeny *of degree prime to* p, $\iota^{\sim}$ is a homomorphism $\mathcal{O}_D \rightarrow \mathrm{End}(A^{\sim})$ such that $tr(x|Lie^*A^{\sim}) = t(x)$ for $x \in \mathcal{O}_D$, $\Lambda^{\sim}$ is a L_0-homogeneous polarization on $A^{\sim}$ which induces the involution * on $\mathcal{O}_D$ and which contains a polarization of degree prime to p, and $\overline{\eta}^{\sim}$ is an equivalence class for the action of K^p of $\mathcal{O}_D \otimes \mathbb{A}^p{}_f$-module isomorphisms $\eta^{\sim}$: $H^1(A^{\sim}, \mathbb{A}^p{}_f) \rightarrow^{\sim} V \otimes \mathbb{A}^p{}_f$ which transform ψ to the form on $H^1(A^{\sim}, \mathbb{A}^p{}_f)$ induced by a polarization in $\Lambda^{\sim}$ up to multiplication by an element of $L_0 \otimes \mathbb{A}^p{}_f$. An isogeny from $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \overline{\eta^{\sim}})$ to $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime}, \overline{\eta}^{\sim\prime})$ is an isogeny from $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim})$ to $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime})$ - an isogeny of degree prime to p is an isomorphism. The class decomposition of $S_{\wp}(K)(\bar{\kappa})$ is in our special case the isogeny classes.

The proof falls into two parts. In the first part it is proved that the set of equialence classes of permissible homomorphisms $\varphi: \mathscr{L} \to G$ parametrize the set of isogeny classes in $S_{\wp}(K)(\bar{\kappa})$. In the second part it is proved that an isogeny class has the decripted structure. The first part will be presented in two variants. The first builts on some unproved conjectures from the algebraic geometry, the second do not need any unproved conjectures but instead a theorem of Kottwitz (which was unproved at the time LR was published but which is now proved (by Kottwitz (unpublished) and independently by Reimann and Zink (RZ))).

The first variant can be outlined in the following way:

By using the *Grothendieck standard conjectures* we can construct the Tannakian category $M_{\bar{\kappa}}$ (over $\mathbb{Q}$) of (all) motives over $\bar{\kappa}$. We can (without use of unproved results) construct the neutral Tannakian category $M_{\bar{\mathbb{Q}}}$ (over $\mathbb{Q}$) of (all) motives over $\bar{\mathbb{Q}}$, the associated affine $\mathbb{Q}$-group is the connected motivic Galois group G^0 (we have chosen an imbedding $\bar{\mathbb{Q}} \to \mathbb{C}$). A sub-Tannakian category $CM_{\bar{\mathbb{Q}}}$ of $M_{\bar{\mathbb{Q}}}$ is generated by the abelian varieties over $\mathbb{C}$ with complex multiplication and the Tate object, the associated affine $\mathbb{Q}$-group is the connected Serre group S. We therefore have a projection $G^0 \to S$. Any abelian variety over $\mathbb{C}$ with complex multiplication can be reduced modulo p (we have chosen an imbedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ determining $\wp$) and the reduced variety determines a motive in $M_{\bar{\kappa}}$. By using the *Hodge conjecture for abelian varteties over* $\mathbb{C}$ *with complex multiplication* we can extend this operation to a functor $CM_{\bar{\mathbb{Q}}} \to M_{\bar{\kappa}}$. If $L \subset \bar{\mathbb{Q}}$ is a CM-field and

${}^{L}CM_{\bar{\mathbb{Q}}}$ is the sub-Tannakian category of $CM_{\bar{\mathbb{Q}}}$ generated by the abelian varieties over $\mathbb{C}$ with complex multiplication through L and the Tate object, then the associated affine $\mathbb{Q}$-group is ${}^{L}S$, and if we let ${}^{L}M_{\bar{\kappa}}$ denote the sub-Tannakian category of $M_{\bar{\kappa}}$ generated by the image of ${}^{L}CM_{\bar{\mathbb{Q}}}$ by the reduction functor, then ${}^{L}M_{\bar{\kappa}}$ is algebraic and by using the *Tate conjecture over a finite field,* we can prove that "the" gerb associated to ${}^{L}M_{\bar{\kappa}}$ is $\wp^{L}$ (constructed in LR and in the appendix, we have a homomorphism $\mathscr{L} \to \wp$). We therefore have an injective homomorphism of gerbs $\wp^{L} \to G_{LS}$ (determined up to conjugation by an element of $\wp^{L}(\bar{\mathbb{Q}})$).

Now let $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \bar{\eta}^{\sim})$ be a point of $S_{\wp}(K)(\bar{\kappa})$. To $A^{\sim}$ is associated a motive in $M_{\bar{\kappa}}$ (belonging to ${}^{L}M_{\bar{\kappa}}$ for L sufficiently large), the homogene part of degree 1 of this motive corresponds to a representation of $\wp$. We can assume that the representation space is V, that the action of D on V determined by $\iota^{\sim}$ is the given action and that some polarization $\eta^{\sim} \in \Lambda^{\sim}$ corresponds to ψ on V. Then the representation maps into G and the composition of this homomorphism $\wp \to G$ with $\mathscr{L} \to \wp$ is a permissible homomorphism φ: $\mathscr{L} \to G$ (that φ is permissible is easy seen in the setting of the second variant of the proof below). If we had chosen another $\eta^{\sim} \in \Lambda^{\sim}$, then the new φ would be equivalent to the former, and if $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime}, \bar{\eta}^{\sim\prime})$ is isogene to $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \bar{\eta}^{\sim})$, then the corresponding equivalence class of permissible homomorphisms $\mathscr{L} \to \wp$ is the same. Conversely: a permissible homomorphism φ: $\mathscr{L} \to G$ factorizes through $\mathscr{L} \to \wp$ and gives thus rise to a representation of $\wp$ and so a motive in $M_{\bar{\kappa}}$, this motive is the homogene part of degree 1 of the motive associated to an abelian variety $A^{\sim}$ over $\bar{\kappa}$, the action of D on the representation space V of φ determines an action $\iota^{\sim}$ of $\mathcal{O}_D$ on

$A^{\sim}$, and the form ψ on V determines a L_0-homogeneous polarization $\overline{\Lambda}^{\sim}$ on $A^{\sim}$, finally there exists a level structure $\overline{\eta}^{\sim}$ on $A^{\sim}$ (because φ is permissible). Thus we have constructed a point $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \overline{\eta}^{\sim})$ of $S_{\wp}(K)(\overline{\kappa})$, another choise of φ (equivalent to the former) would lead to an isogene point of $S_{\wp}(K)(\overline{\kappa})$. These two maps between the set of equivalence classes of permissible homomorphisms $\varphi: \mathscr{L} \rightarrow G$ and the set of isogeny classes of $S_{\wp}(K)(\overline{\kappa})$ are the inverse of each other.

Then we come to the second variant.

A *special point* of $S(K)(\mathbb{C})$ is a triple (T, h, g), where T is a Cartan subgroup of G, $h \in X_\infty$ and factorizes through T and $g \in G(\mathbb{A}_f)$ (two triples are equivalent (and identified) if they differ by action of $G(\mathbb{Q})$ on the left and action of K on the right of g). In the above correspondance between points of $S(K)(\mathbb{C})$ and abelian varieties with additional structures, a special point corresponds to a sixtubel $(A, \iota, \Lambda, \overline{\eta}, R, \theta)$ (up to isomorphism), where the quadrupel $(A, \iota, \Lambda, \overline{\eta})$ corresponds to the point {(h, g)} and R is the CM-algebra (= product of CM-fields) defining T (thus $T(\mathbb{Q}) = \{r \in R^\times \mid r \cdot \overline{r} \in L_0\}$ and $\dim_{\mathbb{Q}} R = \dim_{\mathbb{Q}}(V)/d$) and θ is a complex multiplication through R on (A, ι, Λ,) - that is, an involution preversing imbedding $R \rightarrow End_D(A)$. This sixtubel can be constructed as follows: $\mu_h: \mathbb{C}^\times \rightarrow (R \otimes \mathbb{C})^\times$ determines a complex multiplication (R, Φ), if B is the (polarizable) abelian variety over $\mathbb{C}$ up to isogeny with complex multiplication (R, Φ), we take $A = B^d$. Because $D \otimes_L R = M_d(R)$, D acts on A, this is ι. The representation space of the representation of $^K S$ (K sufficiently large field) corresponding to A (or rather, to the homogeneous part of degree 1) can be identified with V such that the action of D defined by ι is the given action and the "diagonal" action of R on V is that of T. We let Λ be the L_0-homogeneous polarization on A defined by ψ, and $\overline{\eta}$ be the set of isomorphisms $H^1(A^\sim, \mathbb{A}_f) = V\otimes\mathbb{A}_f \xrightarrow{\sim} V\otimes\mathbb{A}_f$ given by Kg^{-1} and we let θ be the "diagonal" action of R on A. If we reduce $(A, \iota, \Lambda, \overline{\eta}, R, \theta)$ modulo p we get a special point $(A^\sim, \iota^\sim, \Lambda^\sim, \overline{\eta}^\sim, R, \theta^\sim)$ of $S_\wp(K)(\overline{\kappa})$.

The second variant can be outlined in the following way:

Given (T, h), if we choose a $g \in G(\mathbb{A}_f)$, then to (T, h, g) we have constructed a special point $(A, \iota, \Lambda, \overline{\eta}, R, \theta)$ of $S(K)(\mathbb{C})$ and (by reduction modulo p) a special point $(A^\sim, \iota^\sim, \Lambda^\sim, \overline{\eta}^\sim, R, \theta^\sim)$ of $S_\wp(K)(\overline{\kappa})$, the isogeny class of $S_\wp(K)(\overline{\kappa})$ containing the point $(A^\sim, \iota^\sim, \Lambda^\sim, \overline{\eta}^\sim)$ is independent of the choise of g. The isogeny classes of $S_\wp(K)(\overline{\kappa})$ constructed from (T, h) and (T', h') are equal if and only if $\psi_{T,\mu h}$ and $\psi_{T',\mu h'}$ (see appendix) are equivalent. This is a consequence of the fact that the existence of an isogeny from $(A^\sim, \iota^\sim, \Lambda^\sim)$ to $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime})$ is equivalent to the existence of an automorphism g of $V\otimes\overline{\mathbb{Q}}$ satisfying the conditions (we have here identified $H^1(A, \mathbb{Q})$ and V in such a way that ι corresponds to the given action of D on V and that the bilinear form ψ_λ on $H^1(A, \mathbb{Q})$ associated to some $\lambda \in \Lambda$ corresponds to ψ, and analogous for A'):

1) g commutes with the action of D

2) g transforms Λ' to Λ

3) if we identify the contravariant rational Dieudonné module associated to $A^\sim$ resp. $A^{\sim\prime}$ with $V\otimes\kappa$, where the F-translation is given by $x \to b^\sim\sigma(x)$ resp. $x \to b^{\sim\prime}\sigma(x)$, with $b^\sim = \chi(b^\sim_0)$ resp. $b^{\sim\prime} = \chi'(b^\sim_0)$ for $b^\sim_0 \in {}^K S(\kappa)$, then we can choose $s \in T(\overline{\mathbb{Q}}_p)$ such that $\overline{g} = gs \in G(\mathbb{Q}_p^{un})$ and $b^{\sim\prime} = \overline{g}b^\sim\sigma(\overline{g})^{-1}$ (for χ and χ' see below)

4) if we identify the ℓ-adic ($\ell \neq p$) cohomology spaces associated to $A^{\sim}$ and $A^{\sim\prime}$ with inner forms of $V \otimes \mathbb{Q}_\ell$, then g shall transform these spaces to each other

5) if the Frobenius endomorphisms on $A^{\sim}$ and $A^{\sim\prime}$ over κ^j (for j sufficiently large) correspond to the automorphisms $\varepsilon^{\sim}$ and $\varepsilon^{\sim\prime}$ on V, then we shall have $\varepsilon^{\sim\prime} = g\varepsilon^{\sim}g^{-1}$ (for j sufficiently large)

these conditions for g are equivalent to the conditions:

1') $g \in G(\overline{\mathbb{Q}})$

2') g is an equialence for the two homomorphism (*) on *the kernel*

$$\wp \xrightarrow{\psi\mu 0} \mathcal{G}_{KS} \xrightarrow{\chi,\chi'} \mathcal{G}_T,\ \mathcal{G}_{T'} \subset \mathcal{G}\ (*)$$

here μ_0 is the canonical cocharacter of ${}^{K}S$, $\psi_{\mu 0}$ is defined in the appendix and the homomorphisms χ: ${}^{K}S \to T$ and χ': ${}^{K}S \to T'$ are defined over $\mathbb{Q}$ and map μ_0 to μ_h and $\mu_{h'}$

3') g is a locally equivalence for the two homomorphisms (*) w.r.t. ζ_∞: $\mathcal{W} \to P$, ζ_p: $\mathcal{D} \to \wp$ and ζ_ℓ: $\mathcal{G}_\ell \to \wp$ (for $\ell \neq p$).

[Sketch of proof: 2') follows from 5) and the definition of ψ_μ. 2) is tantamount to $\psi(gx, gy) = \psi(ax, y)$ for some $a \in L_0 \otimes \overline{\mathbb{Q}}$, and $h'(i)\times\iota = g(h(i)\times\iota)g^{-1}$, but since $v(h_0(i)\times\iota)v^{-1} = \mu_0(-1)\times\iota = (\psi_{\mu 0}\circ\zeta_\infty)(\tau)$, where v = (say) $(\mu_0+\overline{\mu}_0)(\sqrt{i})$, we have $\psi_{\mu h'}\circ\zeta_\infty = ad(g)\circ(\psi_{\mu h}\cdot\zeta_\infty)$. If $b_0^{\sim} \in {}^{K}S(\kappa)$ determines the F-translation and $b_0 \in {}^{K}S(\kappa)$ is constructed from $\psi_{\mu 0}\circ\zeta_p$: $\mathcal{D} \to \mathcal{G}_{KS}$ (as in 1.2), then the *theorem of Kottwitz* states that $b_0 = u_0 b_0^{\sim}\sigma(u_0)^{-1}$ for some $u_0 \in {}^{K}S(\kappa)$, in fact $u_0 \in \mathrm{Im}\ \psi_{\mu 0}(P(\kappa))$, we therefore have $b = ub^{\sim}\sigma(u)^{-1}$, $b' = u'b'^{\sim}\sigma(u')^{-1}$ and $u' = \overline{g}u\overline{g}^{-1}$ ($u = \chi(u_0)$, ...), the condition $b'^{\sim} = \overline{g}b^{\sim}\sigma(\overline{g})^{-1}$

is then equivalent to $b' = \bar{g}b\sigma(\bar{g})^{-1}$, this implies that b' also can be constructed from $ad(g)\circ(\psi_{\mu h}\cdot\zeta_p)$, therefore we must have $\psi_{\mu h'}\circ\zeta_p = ad(g)\circ(\psi_{\mu h}\cdot\zeta_p)$. The above mentioned forms of $V\otimes\mathbb{Q}_\ell$ are determined by a homomorphism ζ_ℓ': $G_\ell \rightarrow \wp$ (a trivialization) and this is equivalent to ζ_ℓ: $G_\ell \rightarrow \wp$, 4) is tantamount to $\psi_{\mu h'}\circ\zeta_\ell' = ad(g)\circ(\psi_{\mu h}\circ\zeta_\ell')$, but this condition is equivalent to $\psi_{\mu h'}\circ\zeta_\ell = ad(g)\circ(\psi_{\mu h}\circ\zeta_\ell)$ (because $\psi_{\mu h'}\circ\zeta_\ell = ad(y')\circ(\psi_{\mu h'}\circ\zeta_\ell') = ad(y')\circ ad(g)\circ(\psi_{\mu h}\circ\zeta_\ell') = ad(g)\circ ad(y)(\psi_{\mu h}\circ\zeta_\ell') = ad(g)\circ(\psi_{\mu h}\circ\zeta_\ell')$, here $y = \psi_{\mu h}(\gamma)$ and $y' = \psi_{\mu h'}(\gamma)$ if $\zeta_\ell = ad(\gamma)\circ\zeta_\ell'$ for $\gamma \in P(\overline{\mathbb{Q}}_\ell)$).]

Now we shall use that two homomorphisms ψ, ψ': $\wp \rightarrow G$ are equal if they are equal on the kernel and locally equal, and that the two homomorphisms (*) composed with the homomorphism $\mathcal{L} \rightarrow \wp$ are $\psi_{T,\mu h}$ and $\psi_{T',\mu h'}$.

Every permissible homomorphism φ: $\mathcal{L} \rightarrow G$ is equivalent to one of the form $\psi_{T,\mu h}$ (LR, Satz 5.3), we can consequently define an injective map from the set of equivalence classes of permissible homomorphisms φ: $\mathcal{L} \rightarrow G$ to the set of isogeny classes of $S_{\wp}(K)(\bar{\kappa})$. This map is surjective because every point $(A^\sim, \iota^\sim, \Lambda^\sim, \bar{\eta}^\sim)$ of $S_{\wp}(K)(\bar{\kappa})$ is component of a special point $(A^\sim, \iota^\sim, \Lambda^\sim, \bar{\eta}^\sim, R, \theta^\sim)$ for some R and $\theta^\sim$ (because $A^\sim$ is defined over a finite field), and a special point of $S_{\wp}(K)(\bar{\kappa})$ is the reduction modulo p of a special point of $S(K)(\mathbb{C})$ (Z2, § 4.4).

Now we come to the second part of the proof.

Let φ: $\mathcal{L} \rightarrow G$ be a permissible homomorphism and let $A \subset S_{\wp}(K)(\bar{\kappa})$ be the corresponding isogeny class, then we shall construct a bijection $A \xrightarrow{\sim} I_\varphi\backslash(X_p\times X^p)/K^p)$ such that the Frobenius action (over κ) on A corresponds to the action $\Phi = (b\times\sigma)^r$ on X_p. We can assume that $\varphi = \psi_{T,\mu h}$, and we choose a $g \in G(\mathbb{A}_f)$. To (T, h, g) we have constructed a special point $(A^\sim, \iota^\sim, \Lambda^\sim, \eta^\sim, R, \theta^\sim)$ of $S(K)(\mathbb{C})$ and (by reduction modulo p) a special point $(A^\sim, \iota^\sim, \Lambda^\sim, \overline{\eta}^\sim, R, \theta^\sim)$ of $S_{\wp}(K)(\overline{\kappa})$. $\mathcal{A}$ is the isogeny class containing $(A^\sim, \iota^\sim, \Lambda^\sim)$. We identify the contravariant rational Dieudonné module of A with $V\otimes\kappa$ as above, then the F-translation is given by $x \to b^\sim\sigma(x)$, where $b^\sim \in T(\kappa)$, furthermore $b^\sim = u^{-1}b\sigma(u)$, where b is constructed from φ (as in 1.2) and $u \in T(\kappa)$. In the first variant this follows from the fact that $\mathcal{D}$ is the gerb associated to the Tannakian category of iso-crystals over κ, and that the association of the contravariant rational Dieudonné module to a motive in $M_{\overline{\kappa}}$ corresponds to the operation of composing a representation of $\wp$ with a homomorphism $\mathcal{D} \to \wp$ which is equivalent to $\zeta_p: \mathcal{D} \to \wp$ (LR, p. 162), and in the second variant this is the meaning of the mentioned theorem of Kottwitz.

If $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime}, \overline{\eta}^{\sim\prime}) \in \mathcal{A}$ and if α is an isogeny from $(A^\sim, \iota^\sim, \Lambda^\sim)$ to $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime})$, then we can construct an element $(x_p, x^p) \in X_p\times X^p/K^p$ as follows: α is the composite of an isogeny α_p whose degree is divisible by p and an isogeny α^p whose degree is prime to p, α_p induces a homomorphism from the contravariant Dieudonné module of $A^{\sim\prime}$ into $V\otimes\kappa$, let M' be the image of this, then M' is a lattice of $V\otimes\kappa$ and $M' = g(V_{\mathbb{Z}}\otimes\mathcal{O}_\kappa)$ for some $g \in G(\kappa)$. If we take $x_p = ugx_0 \in G(\kappa)x_0$ (see 1.2), then $x \in X_p$. α^p is in fact an isomorphism between $(A^\sim, \iota^\sim, \Lambda^\sim)$ and $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime})$, and since $\overline{\eta}^{\sim\prime}$ can be regarded as an element of X^p/K^p, $\overline{\eta}^{\sim\prime}$ determines an element x^p of X^p/K^p. The class of (x_p, x^p) in $I_\varphi\backslash(X_p\times X^p)/K^p)$ is independent of the choise of α, and the map $\mathcal{A} \to I_\varphi\backslash(X_p\times X^p)/K^p)$ is a bijection (remark that we have an isomorphism $I_\varphi \xrightarrow{\sim} \mathrm{Aut}(A^\sim, \iota^\sim, \Lambda^\sim)$ and that u determines an isomorphism $J_\varphi' \xrightarrow{\sim} \mathrm{Aut}(V\otimes\kappa, \iota,$

$\{\psi\}$).

The Frobenius action (over κ) on A is given by $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime}, \bar{\eta}^{\sim\prime}) \to (A^{\sim\prime(q)}, \iota^{\sim\prime(q)}, \Lambda^{\sim\prime(q)}, \bar{\eta}^{\sim\prime(q)})$ (the inverse image by the Frobenius over κ) and if we as isogeny from $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime})$ to $(A^{\sim\prime(q)}, \iota^{\sim\prime(q)}, \Lambda^{\sim\prime(q)})$ choose α (composed with the Frobenius isogeny from $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime})$ to $(A^{\sim\prime(q)}, \iota^{\sim\prime(q)}, \Lambda^{\sim\prime(q)})$, then the lattice of $V\otimes\kappa$ associated to $A^{\sim\prime(q)}$ is the image of M' by the r-th power of the F-translation, that is $(b^{\sim}\times\sigma)^r M'$, and the element X^p/K^p associated to $\bar{\eta}^{\sim\prime(q)}$ is (by the definition of $\bar{\eta}^{\sim\prime(q)}$) that associated to $\bar{\eta}^{\sim\prime}$. The Frobenius action on A, is therefore given by the action of $\Phi = (b^{\sim}\times\sigma)^r$ on X^p.

This bijection between the set of equivalence classes of permissible homomorphisms φ: $\mathscr{L} \to G$ and the set of isogeny classes of $S_{\wp}(K)(\kappa)$ can be refined to a bijection between the set of equivalence classes of j-K-permissible pairs (φ, ε) and the set of j-isogeny classes of $S_{\wp}(K)(\kappa^j)$. A j-permissible pair (φ, ε) is j-K-*permissible* if $(I_\varphi)_\varepsilon\backslash(Y_p{}^j\times Y^p)$ (see 1.2) is non-empty, that is, if

1) $\exists x \in X_p$: $\varepsilon' x = \Phi^j x$

2) $\exists y \in X^p$: $y^{-1}\varepsilon y \in K^p$ (see 1.3).

Two j-K-permissible pairs (φ, ε) and (φ', ε') are equivalent if $\varphi' = ad(g)\circ\varphi$ and $\varepsilon' = ad(g)(\varepsilon)\circ z$ for some $g \in \underline{G}(\mathbb{Q})$ and $z \in Z(\mathbb{Q})_K$. If $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \bar{\eta}^{\sim})$ and $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime}, \bar{\eta}^{\sim\prime})$ belongs to $S_{\wp}(K)(\kappa^j)$, then an j-*isogeny* from $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim})$ to $(A^{\sim\prime}, \iota^{\sim\prime}, \Lambda^{\sim\prime})$ is an isogeny which commute with the Frobenius endomorphisms over κ^j on $A^{\sim}$ and $A^{\sim\prime}$. The j-isogeny class corresponding to (φ, ε) is that containing the point $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \bar{\eta}^{\sim})$ of $S_{\wp}(K)(\kappa^j)$ constructed as follows: We can assume that $\varphi = \psi_{T,\mu h}$ and $\varepsilon \in T(\mathbb{Q})$ (LR, Lemma 5.23). Let $v \in T(\overline{\mathbb{Q}}_p)$ and $b \in T(\kappa)$ be constructed from $\varphi\circ\zeta_p$ as in 1.2. Choose $\overline{g}_p \in G(\kappa)$ such that for $x = \overline{g}_p\cdot x_0$ is $\varepsilon' x = \Phi^{j} x$ and $y \in X^p$ such that for $y^{-1}\varepsilon y \in K^p$, and if the F-translation on the contravariant rational Dieudonné module $V\otimes\kappa$ of $A^{\sim}$ (constructed from (T, h)) is given by $x \to b^{\sim}\sigma(x)$ where $b^{\sim} \in T(\kappa)$, choose $u \in T(\kappa)$ such that $b = ub^{\sim}\sigma(u)^{-1}$. Let $g \in G(\mathbb{A}_f)$ be defined by $g = v^{-1}u^{-1}\overline{g}_p$ and $g^p = y$.

To (T, h, g) we have constructed a special point $(A, \iota, \Lambda, \overline{\eta}, R, \theta)$ of $S(K)(\mathbb{C})$ and (by reduction modulo p) a special point $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \overline{\eta}^{\sim}, R, \theta^{\sim})$ of $S_{\wp}(K)(\overline{\kappa})$, $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \overline{\eta}^{\sim})$ belongs to $S_{\wp}(K)(\kappa^j)$ and the j-isogeny class of $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim}, \overline{\eta}^{\sim})$ is independent of the choises. The lattice $L = g\cdot V_{\mathbb{Z}}$ of V (and the complex structure on $V\otimes\mathbb{R}$ given by h) define an abelian variety A_0 over $\mathbb{C}$ in the isogeny class of (A, ι, Λ) (namely $A_0 = ((V\otimes\mathbb{R})/L)^*$), and since $\varepsilon \in G(\mathbb{Q})$ and $\varepsilon L \subset L$, ε defines an isogeny on (A_0, ι, Λ) and the reduction of this to $(A^{\sim}, \iota^{\sim}, \Lambda^{\sim})$ is the Frobenius endomorphism over κ^j.

The above bijection between the isogeny class corresponding to φ and $I_\varphi\backslash(X_p\times X^p)/K^p)$ has in the present setting as analogous a bijection between the j-isogeny class $\mathcal{A}$ corresponding to (φ, ε) and $(I_\varphi)_\varepsilon\backslash(Y_p^j\times Y^p)$: if in the above proof we choose α such that it transforms the Frobenius endomorphism (over κ^j) on $A^{\sim}{}'$ to $\varepsilon \in I_\varphi$, then x_p belongs to $Y^j_p \subset X_p$ and x^p belongs to $Y^p \subset X^p/K^p$, the class of (x_p, x^p) in $(I_\varphi)_\varepsilon\backslash(Y_p^j\times Y^p)$ is independent of the choice of α, and the map $\mathcal{A} \to (I_\varphi)_\varepsilon\backslash(Y_p^j\times Y^p)$ is a bijection.

A j-*triple* $(\varepsilon, \delta, \gamma)$ consist of $\varepsilon \in G(\mathbb{Q})_{s.s.}$ which is elliptic at infinety, a $\delta \in G(F^n)$ $(n = jr)$ such that $Nm_{F^n/\mathbb{Q}_p}\delta$ is stably conjugate to ε and $\gamma_\ell \in G(\mathbb{Q}_\ell)$ (for each $\ell \neq p$) such that γ_ℓ is stably conjugate to ε (and conjugate to ε for almost all ℓ). The j-triples $(\varepsilon, \delta, \gamma)$ and $(\varepsilon', \delta', \gamma')$ are equivalent if ε and ε' are stably conjugate, δ and δ' are $G(F^n)$-σ-conjugate, and γ and γ' are conjugate, and they are K-equivalent if (ε', δ', γ') is equivalent to (εz, εw, γz), where $z \in Z(\mathbb{Q})_K$ and $w \in Z(F^n) \cap K_p(\mathcal{O}_{F^n})$ satisfies $Nm_{F^n/\mathbb{Q}_p} w = z$. We will not distinguish between a j-triple and its equivalence class.

The *Kottwitz invariant* of a j-triple (ε, δ, γ) is the element $\beta(\delta, \gamma) \in K(G_\varepsilon/\mathbb{Q})^D$ (see 1.7) - if $\varepsilon \sim_K \varepsilon'$ (stable conjugacy modulo $Z(\mathbb{Q})_K$) we can identify $K(G_\varepsilon/\mathbb{Q})^D$ and $K(G_{\varepsilon'}/\mathbb{Q})^D$, and K-equivalent (ε, δ, γ) and (ε', δ', γ') have equal Kottwitz invariants (LR, Lemma 5,18).

To an equivalence class of j-permissible pairs $(\varphi, \bar{\varepsilon})$ we have (in 1.3) constructed an equivalence class of j-triples (ε, δ, γ). The Kottwitz invariant of a such j-triple is 1, and conversely: any j-triple whose Kottwitz invariant is 1 is the j-triple of a j-permissible pair (LR, Satz 5,25), precise i(ε) inequivalent j-permissible pairs have the same equivalence class of j-triples (ε, δ, γ). Therefore we can to every j-isogeny class *A* of $S_{\wp}(K)(\kappa^j)$ associate a K-equivalence class of j-triples (ε, δ, γ), namely that associated to the equialence class of j-K-permissible pairs corresponding to *A*. The K-equivalence of j-triples of the j-isogeny class containing $(A^\sim, \iota^\sim, \Lambda^\sim, \bar{\eta}^\sim) \in S_{\wp}(K)(\kappa^j)$ can be constructed directly as follows: The Frobenius endomorphism on $A^\sim$ (over κ^j) determines an automorphism $\varepsilon^\sim$ of $V \otimes \bar{\mathbb{Q}}$, it belongs to $G(\bar{\mathbb{Q}})$ and can be chosen conjugate to an element $\varepsilon \in G(\mathbb{Q})_{s.s.}$. If the F-translation on the contravariant rational Dieudonné module $V \otimes \kappa$ of $A^\sim$ is given by $x \to \varepsilon^\sim \sigma(x)$ ($b^\sim \in G(\kappa)$), then $b^\sim = \varepsilon^\sim b^\sim \sigma(\varepsilon^\sim)^{-1}$ (remark that $\varepsilon^\sim \in G(\mathbb{Q}_p^{un})$ because it is conjugate to ε) and we must have $Nm_{F^n/\mathbb{Q}_p} b^\sim = \varepsilon^\sim c^{-1} \sigma^n(c)$ for some $c \in G(\kappa)$, we take $\delta = c b^\sim \sigma(c)^{-1}$ (then $\delta \in G(F^n)$). Finally the Frobenius endomorphism (over κ^j) on $A^\sim$ determines via a $\eta^\sim \in \overline{\eta}^\sim$ an automorphism γ_ℓ of $V\otimes\mathbb{Q}_\ell$ (for $\ell \neq p$), this belongs to $G(\mathbb{Q}_\ell)$ (in fact it is conjugate to an element of K_ℓ).

A long step toward a proof of the conjecture in the general case would have been taken if we to every point of $S_{\wp}(K)(\kappa^j)$ can construct a K-equivalence class of j-triples and prove that its Kottwitz invariant is l.

3.2 Let G be an unramified connected reductive $\mathbb{Q}_p$-group (such p that G_{der} is simply connected), let K be a hyperspecial subgroup and let F be an unramified extension of $\mathbb{Q}_p$ of degree n.

Let $\overline{M}$ be a G(F)-conjugacy class of homomorphisms $G_m \rightarrow G_F$ such that one (and hence all) of the representations G_m on $\mathrm{Lie}(G_{\overline{\mathbb{Q}}p})$ constructed from homomorphisms in $\overline{M}$ has no other weights than 0, ±1. Let $f^\sim \in \mathcal{H}(G(F), K(\mathcal{O}_F)))$ be the characteristic function of the coset in $K(\mathcal{O}_F)\backslash G(F)/ K(\mathcal{O}_F)$ corresponding to $\overline{M}$ (see 1.2) and let f $\in \mathcal{H}(G(\mathbb{Q}_p), K(\mathbb{Z}_p)))$ be the image of $f^\sim$ by the base-change homomorphism (characterized by the property that tr $\pi_\varphi(f) = \mathrm{tr}\ \pi_{\varphi'}(f^\sim)$, where $\varphi' = \varphi'|_{\mathrm{Gal}(\mathbb{Q}p^{un}/F)}$ for every admissible homomorphism φ: $\mathrm{Gal}(\mathbb{Q}_p^{un}/F) \rightarrow {}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$.

If $\varepsilon \in G(\mathbb{Q}_p)^n$ (defined as in 1.4 but w.r.t. M), let T be an elliptic Cartan subgroup of G_ε and let $\mu \in X_*(T)$ be M_ε-conjugate to a μ satisfying the condition 1.4, then the element $b_\varepsilon \in T(\kappa)$ constructed from the homomorphisms $\xi_{-\mu}$: $\mathcal{D} \rightarrow T(\overline{\mathbb{Q}}_p)\times\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (see 1.7) satisfies $\mathrm{Nm}_{F/\mathbb{Q}p}b_\varepsilon = \varepsilon c^{-1}\sigma^n(c)$ ($c \in G(\kappa)$), and if $\delta_\varepsilon = cb_\varepsilon\sigma(c)^{-1}$ then $\delta_\varepsilon \in G(F)$ (and $\mathrm{Nm}_{F/\mathbb{Q}p}\delta_\varepsilon = c\varepsilon c^{-1}$) and we have

$$c(G_\varepsilon)\cdot O(\varepsilon, f) = c(G^\sigma_{\delta\varepsilon})\cdot TO(\delta_\varepsilon, f^\sim) \quad (*)$$

($G^\sigma_{\delta\varepsilon}$ is the inner form of G_ε, this allows us to choose compatible measures on $G^{\sigma}_{\delta\varepsilon}(\mathbb{Q}_p)$ and $G_{\varepsilon}(\mathbb{Q}_p)$).

If $\varepsilon \in G(\mathbb{Q}_p)_{s.s.} \backslash G(\mathbb{Q}_p)^n$ then $O(\varepsilon, f) = 0$.

(*) is proved in K7 for M trivial (that is, $\tilde{f}$ and f the unit elements) and in AC for G = GK(n) and *arbitrary* $\tilde{f} \in \mathcal{H}(G(F), K(\mathcal{O}_F))$, $\varepsilon \in G(\mathbb{Q}_p)_{s.s.}$ and $\delta \in G(F)$ such that ε is conjugate (in G(F)) to $Nm_{F/\mathbb{Q}_p}\delta$ - in fact, this result is conjectured true for general G if orbital- resp. twisted orbital integral is replaced by stable orbital resp. stable twisted orbital integral - in this case $SO(\varepsilon, f) = 0$ if $\varepsilon \in G(\mathbb{Q}_p)_{s.s.}$ and not conjugate to a $Nm_{F/\mathbb{Q}_p}\delta$.

3.3 Let G be as in this paper and let $(H, s, \eta) \in \mathcal{E}$. For $\gamma \in H(\mathbb{Q})_{e,(G,H)\text{-reg}}$ and $\varepsilon \in G(\mathbb{Q})_e$ such that γ is the image of ε, we have

$$i(\gamma)\cdot|K(H_\gamma/\mathbb{Q})|^{-1}\,\tau(H_\gamma)\cdot\tau(H)^{-1} = i(\varepsilon)\cdot|K(G_\varepsilon/\mathbb{Q})|^{-1}\cdot\tau(G_\varepsilon)\cdot\tau(G)^{-1}.$$

$\tau(H_\gamma)$ and $\tau(G_\varepsilon)$ are as defined in 1.6 and the measures on $H_\gamma(\mathbb{A})$ and $G_\varepsilon(\mathbb{A})$ are chosen compatible (recall that H_γ is an inner form of G_ε) - this measure on $H_\gamma(\mathbb{A})$ (and an arbitrary measure on $H(\mathbb{A})$) is used to define orbital integral on H. $\tau(H)$ and $\tau(G)$ are the Tamagawa numbers (proved in K6 for regular elements - we have used that Kottwitz in K8 has proved that $\tau(G) = 1$ for G simply connected semi-simple (if G has no E8 factor)).

3.4 Let G be a connected reductive $\mathbb{R}$-group (such that G_{der} is simply connected) which has discrete series representations, let T be a fundamental Cartan subgroup and let ξ be a rational representation of G. For each $\varepsilon \in G_\varepsilon(\mathbb{R})$ we choose a measure on $G_\varepsilon(\mathbb{R})$ such that the measures on $G_\varepsilon(\mathbb{R})$ and $G_{\varepsilon'}(\mathbb{R})$ are compatible ε and ε' are stably conjugate - then we have a measure on the compact (modulo

$Z(\mathbb{R})$) inner form $G_\varepsilon'(\mathbb{R})$ of $G_\varepsilon(\mathbb{R})$. We define $\alpha: G(\mathbb{R}) \to \mathbb{R}$ by

$$\alpha(\varepsilon) = \begin{cases} c(G_\varepsilon') \operatorname{tr} \xi(\varepsilon)/\mathrm{meas}(Z(\mathbb{R})\backslash G_\varepsilon'(\mathbb{R})) & \text{if } \varepsilon \in G(\mathbb{R})_e \\ 0 & \text{if } \varepsilon \in G(\mathbb{R})\backslash G(\mathbb{R})_e. \end{cases}$$

If ε' is stably conjugate to ε, then $\alpha(\varepsilon') = \alpha(\varepsilon)$.

Let (H,s,η) be an endoscopic datum for G (we assume that $\eta(s) \in {}^LT^0$) for which there is an isomorphism $X_*(T) \leftrightarrow X^*({}^LT^0)$ such that this, the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on T and $\eta(s)$ determine (H, s, η), and let us choose an extension η': ${}^LH^0\times W_{\mathbb{R}} \to {}^0G^0\times W_{\mathbb{R}}$ of η and a transfer factor $\Delta(\ ,\ )$.

There exist a function $f^H{}_\xi$ on $H(\mathbb{R})$ such that

$$\Delta(\gamma, f^H{}_\xi) = \begin{cases} \Delta(\gamma, \varepsilon)\cdot\alpha(\varepsilon) & \text{if } \gamma \in H(\mathbb{R})_e \\ 0 & \text{if } \gamma \in H(\mathbb{R})_{s.s.}\backslash H(\mathbb{R}), \end{cases}$$

here $\varepsilon \in T(\mathbb{R})$ is chosen such that γ is the image of ε via the isomorphism $X_*(T) \leftrightarrow X^*({}^LT^0)$ (obvius for H_γ an elliptic Cartan subgroup of G, and proved in L7, §6 and Ca for H = GL(2) and G an inner form of H) (the measure on $H(\mathbb{R})$ is of course that compatible with the measure on $G_\varepsilon(\mathbb{R})$ - H_γ is an inner form of G_ε).

If (H,s,η) is not elliptic we take $f^H{}_\xi = 0$.

3.5 Let G be as in 3.2, let (H, s, η) be an endoscopic datum for G and let $\varphi \in \mathscr{H}(G(\mathbb{Q}_p), K)$ be the characteristic function of K.

If there exists $\gamma \in H(\mathbb{Q}_p)_{s.s.(G,H)\text{-reg}}$ such that the sum below is non-zero, then H is unramified (proved in LL for H elliptic Cartan subgroup of G = GL(2)). We choose an extension η': ${}^LH^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p) \to {}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ of η, and we can choose a hyperspecial subgroup K^H of $H(\mathbb{Q}_p)$

such that every $\gamma \in K^H$ is the image of a $\varepsilon \in K$.

There exists a function $\varphi^H \in \mathscr{H}(H(\mathbb{Q}_p), K)$ such that if $\rho \in H(\mathbb{Q}_p)_{s.s.(G, H)\text{-reg}}$ then

$$SO(\gamma, \varphi^H) = \begin{cases} \Delta(\gamma, \varepsilon) \Sigma\, \kappa(\rho)\cdot c(G_{\rho\varepsilon})\cdot O(\rho_\varepsilon, \varphi) \text{ (sum over } \delta \in \mathscr{E}(G_\varepsilon, \mathbb{Q}_p)) \\ \quad \text{if } \gamma \text{ is the image of } \varepsilon \in G(\mathbb{Q}_p)_{s.s.} \\ 0 \text{ if } \gamma \text{ is not the image of any } \varepsilon \end{cases}$$

(see 3.7).

Now we assume that $\eta(s)^m \in Z$ for some m.

Notation:

$\overline{M}$ is a G(F)-conjugacy class of homomorphisms $G_m \to G_F$ such that one (and hence all) of the representations of G_m on $\mathrm{Lie}(G_{\overline{\mathbb{Q}}_p})$ constructed from homomorphisms in $\overline{M}$ has no other weights than 0, ±1.

$\Omega_\mu \subset X^*({}^LT^0)$ is the Weyl-group orbit determined by $\overline{M}$.

${}^0r$ is the (finite dimensional) representation of ${}^LG^0\times \mathrm{Gal}(\mathbb{Q}_p^{un}/F)$ (unique up to equivalence) which is irreducible on ${}^LG^0$ having extreme ${}^LT^0$-weights Ω_μ and for which $\mathrm{Gal}(\mathbb{Q}_p^{un}/F)$ acts trivially on the ${}^LB^0$-highest weight space.

r is ${}^0r$ induced to ${}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$.

$n \in [F:\mathbb{Q}_p]\cdot\mathbb{N}$.

$f \in \mathscr{H}(G(\mathbb{Q}_p), K)$ is associated to the class function $x \to \mathrm{tr}\, r(x^n)$ on ${}^LG^0\times\mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$ by Satake transform.

$\gamma \in H(\mathbb{Q}_p)_{s.s.,(G, H)\text{-}\underline{reg}}$ is the image of $\varepsilon \in G(\mathbb{Q}_p)^n$ (defined as in 1.4 but w.r.t. $\overline{M}$).

A Cartan subgroup T of G_ε and an isomorphism $X_*(T) \leftrightarrow X^*({}^LT^0)$ are chosen such that they arise from the correspondance between γ and ε (see 1.9).

$\mu_0 \in X_*(T)$ is M_ε-conjugate to a μ satisfying the condition in 1.4.

$^0r^H$ is the restriction of ^0r to $^LH^0 \times Gal(\mathbb{Q}_p{}^{un}/F)$ (via η_p).

$\mathcal{H} = \{(\mu - \mu_0)(\eta(s)) \mid \mu \in \Omega_\mu\} \subset$ roots of unity.

For $i \in \mathcal{H}$ $^0r^{H,i}$ is the subrepresentation of $^0r^H$ determined by $\{(\mu - \mu_0)(\eta(s)) \mid \eta(s) = i\}$.

$r^{H,i}$ is $^0r^{H,i}$ induced to $^LH^0 \times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$.

$f^H{}_\gamma \in \mathcal{H}(H(\mathbb{Q}_p), K^H)$ is associated to the class function $x \to \Sigma_{i \in \mathcal{H}}\, i \operatorname{tr} r^{H,i}(x^n)$ on $^LH^0 \times Gal(\mathbb{Q}_p{}^{un}/\mathbb{Q}_p)$ by Satake transform.

Then: $f^H{}_\gamma$ is independent of the choise of ε and we have

$$SO(\gamma, f^H{}_\gamma * \varphi^H) = \Delta(\gamma, \varepsilon)\, \Sigma\, \kappa(\rho) \cdot c(G_{\rho\varepsilon}) \cdot O(\rho_\varepsilon, f*\varphi)$$
$$\text{(sum over } \rho \in \mathscr{E}(G_\varepsilon, \mathbb{Q}_p)).$$

If $\varepsilon \in G(\mathbb{Q}_p)_{s.s.} \backslash G(\mathbb{Q}_p)^n$ then $O(\varepsilon, f) = 0$.

If $\gamma \in H(\mathbb{Q}_p)_{s.s.,(G,H)\text{-reg}}$ is not the image of any $\varepsilon \in G(\mathbb{Q}_p)^n$, then $SO(\gamma, f^H{}_\gamma * \varphi^H) = 0$, here f^H is constructed as above but $\mu_0 \in X_*(^LT^0)$ is chosen arbitrary.

3.6 Let G be as in 3.4. There exists a function f^G on $G(\mathbb{R})$ such that

$$SO(\varepsilon, f^G) = \alpha(\varepsilon)$$

for $\varepsilon \in G(\mathbb{R})_{s.s.}$ (proved in L7, L6 and Ca for G = GL(2)) - the measure on $G(\mathbb{R})_\varepsilon$ is that entering the defintion of α).

Let (H, s, η) be as in 3.4 and let $\psi \in \Phi(H)$ be such that $\varphi = \eta' \circ \psi \in \Phi(G)_e$. We can assume that $\varphi(\mathbb{C}^\times) \subset {}^LT^0 \times \mathbb{C}^\times$ and $\varphi(\tau) = g \times \tau$ where $g \in Norm_{LG0}(^LT^0)$. The action ι' on $^LT^0$ given by $\varphi(\tau)$ corresponds (via $X^*(^LT^0) \leftrightarrow X_*(T)$) to the action on T given by the non-trivial element in $Gal(\mathbb{C}/\mathbb{R})$, therefore $^LT^0 \times Gal(\mathbb{C}/\mathbb{R})$ for this action is the L-group of T. To φ is (by the Langlands correspondance, see Bo) associated a continuous regular character λ_0 of $T(\mathbb{R})$ and so a discrete series reprsentation π_0 of $G(\mathbb{R})$, this belongs to $\Pi(\varphi)$ and we have

$\Sigma_{\pi\in\Pi(\psi)} <1, \pi> \text{tr } \pi(f^H) = e_\infty <\eta(s), \pi_0> \Sigma_{\pi\in\Pi(\psi)} <1, \pi> \text{tr } \pi(f^G)$
(for e_∞ and < , > see 3.7).

We can replace the isomorphism $X_*(T) \leftrightarrow X^*(^LT^0)$ by the composite with a $\omega \in \Omega(^LG^0, {}^LT^0) = \Omega(G(\mathbb{C}), T(\mathbb{C}))$ (because the action of ω on T is defined over $\mathbb{R}$), if we do so we must multiply f^H and $<\eta(s), \pi_0>$ by $\kappa(\{\omega\})$ (= ±1), where κ is the character of $H^1(\mathbb{R}, T) = \pi_0(^LT^{0\Gamma\infty})^D$ determined by $\{\eta(s)\} \in \pi_0(^LT^{0\Gamma\infty})^D$, as we note that $\Omega(G(\mathbb{C}), T(\mathbb{C})/ \Omega(G(\mathbb{R}), T(\mathbb{R}) = \mathcal{D}(T/\mathbb{R}) \subset H^1(\mathbb{R}, T)$ and $<\eta(s), \pi^\omega{}_0> = \kappa(\{\omega\})<\eta(s), \pi_0>$, here $\pi^\omega{}_0$ is attachad to $\lambda_0\circ\omega$.

3.7 Let G be a connected reductive $\mathbb{Q}_\nu$-group (ν place) (such that G_{der} is simply connected) let (H, s, η) be an endoscopic datum for G. Choose an extension $\eta': {}^LH^0\times L_{\mathbb{Q}\nu} \to {}^LG^0\times L_{\mathbb{Q}\nu}$ of η, and choose a transfer factor $\Delta_\nu(\gamma, \varepsilon)$.

There exists an $e_\nu \in \mathbb{C}^\times$ such that the following is true: if the function f on $G(\mathbb{Q}_\nu)$ and f^H on $H(\mathbb{Q}_\nu)$ are connected by

$$SO(\gamma, f^H) = \begin{cases} \Delta_\nu(\gamma, \varepsilon)\, \Sigma\, \kappa(\rho)\cdot c(G_{\rho\varepsilon})\cdot O(\rho_\varepsilon, f) \\ \quad \text{(sum over } \rho \in \mathcal{E}(G_\varepsilon, \mathbb{Q}_\nu)) \\ \quad \text{if } \gamma \text{ is the image of } \varepsilon \in G(\mathbb{Q}_\nu)_{s.s.} \\ 0 \text{ if } \gamma \text{ is not the image of any } \varepsilon \end{cases}$$

(here $\gamma \in H(\mathbb{Q}_\nu)_{s.s.,(G,H)\text{-reg}}$), then we have for each $\psi \in \Phi(H)_{temp}$ that $\varphi = \eta'\circ\psi \in \Phi(G)$:

$$\Sigma_{\pi\in\Pi(\psi)} <1, \pi> \text{tr } \pi(f^H) = e_\nu\, \Sigma_{\pi\in\Pi(\psi)} <\eta(s), \pi> \text{tr } \pi(f),$$

< , > is the usual pairing $\zeta_\varphi\times\Pi(\varphi) \to \mathbb{C}$, where $\zeta_\varphi = S_\varphi/ (S_\varphi)^0Z = \pi_0(S_\varphi/Z)$ and $S_\varphi = \{g \in {}^LG^0 \mid ad(g)\circ\varphi = \varphi\}$, < , > is not canonical, but this does not matter, since the global < , >, which is the product of all the local < , >, is canonical.

For a given function f on $G(\mathbb{Q}_\nu)$ (smooth and of compact support) we can construct a function f^H on $H(\mathbb{Q}_\nu)$ such that f and f^H are connected as above (see LL for G = GL(2), LS2 for G a form of SL(3) and Sh for $\nu = \infty$).

3.8 Let G, f^G be as in 3.6 and let $\varphi \in \Phi(G)_{temp}$. If $\Sigma_{\pi\in\Pi(\varphi)}$ tr $\pi(f^G) \neq 0$, then φ is elliptic and $\Pi(\varphi)$ is the L-packet of discrete series representations of $G(\mathbb{R})$ associated to one of the absolutly irreducible components $\xi^\vee{}_{\Pi(\varphi)}$ of $\xi^\vee$, furthermore we have $\Sigma_{\pi\in\Pi(\varphi)}$ tr $\pi(f^G) = (-1)^d \cdot$ the multiplicity of $\xi^\vee{}_{\Pi(\varphi)}$ in $\xi^\vee$ (this result is used only in the conclusion).

Let G, H, f^H be as in 3.4 and let $\psi \in \Phi(H)_{temp}$. If $\Sigma_{\pi\in\Pi(\psi)}$ tr $\pi(f^G) \neq 0$, then ψ and $\varphi = \eta'\circ\psi$ are elliptic (and so φ is admissible for G).

Let G, H be as in 3.7 and let the function φ^H on $H(\mathbb{Q}_\nu)$ be connected with the characteristic function φ of K (compact open subgroup of $G(\mathbb{Q}_\nu)$). If $\Sigma_{\pi\in\Pi(\psi)} <1, \pi>$ tr $\pi(\varphi^H) \neq 0$, then $\varphi = \eta'\circ\psi$ is admissible for G.

3.9 Let G be as in this paper and let $(H, s, \eta) \in \mathscr{E}$. Let $\bar{\gamma} \in \bar{H}(\mathbb{Q})_{e,(G,H)\text{-reg}}$ and $\bar{\varepsilon} \in G(\mathbb{Q})_e$ be such that $\bar{\gamma}$ is the image of $\bar{\varepsilon}$. Choose the local transfer factors $\Delta_\nu(\ ,\ )$ such that $\Delta_\nu(\bar{\gamma}, \bar{\varepsilon}) = 1$ for almost all places ν and $\Pi_\nu \Delta_\nu(\bar{\gamma}, \bar{\varepsilon}) = 1$. Then $e_\nu = 1$ for almost all places ν and $\Pi_\nu e_\nu = 1$.

3.10 We assume that (a sufficiently large part of) the Langlands correspondance has been constructed - that is, for a given reductive algebraic group, we have a map (having the expected properties) from the equivalence classes of admissible homomorphisms from the Weil- (or rather the Langlands-) group into the L-group associated to the group to the L-packets of representations of the group - the map is a bijection in the local case and maps to automorphic representations in the global case.

Let G be a connected reductive $\mathbb{Q}$-group, let ${}_0Z$ be a closed subgroup of $Z(\mathbb{A})$ of the form $\Pi_{v0} Z_v$ (Z center of G) such that ${}_0ZZ(\mathbb{Q})$ is closed in $Z(\mathbb{A})$ and ${}_0ZZ(\mathbb{Q})\backslash Z(\mathbb{A})$ is compact, let χ be a character of $({}_0Z\cap Z(\mathbb{Q}))\backslash{}_0Z$ and let $\Phi(G)_e$ be the set of (equivalence classes of) elliptic tempered admissible homomorphisms φ: $L_{\mathbb{Q}} \rightarrow {}^LG^0\times L_{\mathbb{Q}}$ such that $\chi_\varphi|_0Z = \chi$ ($L_{\mathbb{Q}}$ is the Langlands group, it is an extension of $W_{\mathbb{Q}}$ by a compact group, see L5 and K3). Then the stable tempered cuspidal part of the trace is (K3)

$$d_\varphi^{-1} \Sigma_{\varphi\in\mathbf{\Phi}(G)e} \Sigma_{\pi\in\Pi(\pi)} n_\pi \operatorname{tr} \pi(f)$$

- d_φ is the number of (global) equivalence classes in the local equivalence class of φ (d_φ different classes of $\Phi(G)_e$ parametrize $\Pi(\varphi)$) and $n_\varphi = d_\varphi^{-1} |\zeta_\varphi|^{-1} <1, \pi>$ is the "stable multiplicity" of π - f is assumed to be of the form $f = \Pi_v f_v$ and to satisfy $f(zg) = \chi(z)^{-1}f(g)$ for $z \in {}_0Z$, and $\pi(f) = \int_{Z\backslash G(\mathbb{A})} \pi(g)f(g)\, dg$) (for all this see LL).

This part of the stable trace is "contained" in the stable elliptic part of the trace.

Conclusion

The fixed prime number p in this paper is assumed to be such that E is unramified at p, K_p is hyperspecial and that S (K) has good reduction at $\wp$ for $\wp|p$. If S(K) has not good reduction at $\wp$, the action of $W_{E\wp}$ on $H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell})$ (via $W_{E\wp} \subset Gal(\overline{E}_\wp/E_\wp)$, see 1.l) need not be unramified (that is, trivial on $I_{E\wp}$ or factorize through $W_{E\wp} \rightarrow$ Gal $(E^{un}{}_\wp/E_\wp) = Gal(\overline{\kappa}/\kappa))$, therefore the action of a Frobenius of $W_{E\wp}$ is not necesserily well defined, but it is on $H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell})^{IW\wp}$, thus the local zeta function of (S(K), ξ) at $\wp$ could be defined by substituting this space in the co-homology formula of l.l.

We expect that all the local zeta functions (as well as the remaining part of (14) for good p) can be expressed in terms of L-functions of a form not very different from that of (14).

The Hasse-Weil zeta function of (S(K), ξ) is the *inverse* product of the local zeta functions at all the finite places of E, and this should thus has an expression in terms of L-functions. However in order to get a more appropriate form of this expression, as well as a more appropriate form of the functional equation which we expect the zeta function to satisfy, we will multiply the Hasse-Weil zeta function by local "zeta functions" also at the infinite places of E. We will define these 1ocal zeta functions such that (14) remains true at infinity. After this we will make a bid for the final form of the expression of the zeta function in terms of L-functions and for the functional equation.

We can get an idea for the definition of the local zeta funtions at infinity by studying the cohomology formula for the local zeta function at a finite place where the reduction is good. We do namely observe that we obtain the same cohomology groups if we first reduce $V(\mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{K/K0} S(K_0) \to S(K)$ modulo $\wp$ - $Gal(\bar{\kappa}/\kappa)$ acts on these cohomology groups and in the formula we interpret $\Phi_\wp$ as the Frobenius in $Gal(\bar{\kappa}/\kappa)$. If we base change $V(\mathbb{Z}/\ell^n\mathbb{Z})\otimes_{K/K0} S(K) \to S(K)$ via an imbedding $\nu: E \to \mathbb{C}$ (an infinite place), the corresponding sheaf over $S_\nu(K)(\mathbb{C})$ (= $(S(K)\otimes_\nu\mathbb{C})(\mathbb{C})$) appears by tensoring by $\mathbb{Z}/\ell^n\mathbb{Z}$ a locally free sheaf of $\mathbb{Z}$-modules over $S_\nu(K)(\mathbb{C})$ (see the final remark in 1.1). If we instead tensorize that sheaf by $\mathbb{Q}$, we get a locally free sheaf of $\mathbb{Q}$-vector spaces $F_{\xi,\nu}(K)$ over $S_\nu(K)(\mathbb{C})$. We can define a representation ρ'_i of $W_{\overline{\nu(E)}}$ on the $\mathbb{Q}$-vector space $H^i(S_\nu(K), F_{\xi,\nu}(K))\otimes_\mathbb{Q}\mathbb{C}$ (rational cohomology) for i = 0, 1, ..., 2d (d = dimS(K)) by letting the action of $\mathbb{C}^\times$ be given by the Hodge structure (that is, by the product of the action $z \to \bar{z}^{-p}\bar{z}^{-q}$ on a subspace of type (p, q) and the action $z \to^{\nu} \xi\circ\tau\, h(z)$ on ${}^\nu V_\mathbb{C}$ in the notation below), if $\nu(E) \subset \mathbb{R}$ the complex conjugation on $S_\nu(K)$ induces an action ι^* on cohomology mapping a subspace of type (p, q) to a subspace of type (q, p), and the action of τ on a such subspace is taken to be $(-1)^p\iota^*$ (or if we like $i^{p+q}\iota^*$). By inducing ρ_i' to $W_\mathbb{R}$ we get a representation ρ_i of $W_\mathbb{R}$. This definition is motivated by the considerations below. The zeta function of (S(K), ξ) at the infinite place ν should now be defined by

$$Z(s, S_\nu(K), F_{\xi,\nu}(K))) = \Pi_{i=1}^{2d} L(s, \rho_i)^{(-1)^\wedge(i+1)}$$

(for the definition of the L-function L(s, ρ) for ρ a representation of $W_\mathbb{R}$ see Ta).

$S_\nu(K)$ is conjectured to be the Shimura variety associated to ${}^\nu G$, ${}^\nu X_\infty$, ${}^\nu K$ defined in the following way: Langlands has constructed an extension of the connected Serre group S^0 (denoted S in 3.1)

$$S^0 \to S \to Gal(\overline{\mathbb{Q}}/\mathbb{Q})$$

with a continuos splitting sp: $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to S(\mathbb{A}_f)$ (see L5 or MSl - the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on S^0 defined by this extension is the algebraic action - S is the Serre group, that is, the $\mathbb{Q}$-rational pro-algebraic group associated to the neutral Tannakian category $CM_{\mathbb{Q}}$ of motives over $\mathbb{Q}$ generated by the abelian varieties over $\mathbb{Q}$ of potential CM-type, the Tate object and the Artin motives (D3) - it is conjectured that for a motive in $CM_{\mathbb{Q}}$ the action of $\tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the cohomology is given by the action (on the representation space) of $sp(\tau) \in S(\mathbb{A}_f)$). For $\tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ the extension defines an element $c(\tau) \in H^1(\mathbb{Q}, S^0)$ (by $\sigma \to a^{-1}\sigma(a)$ if $a \in S(\overline{\mathbb{Q}})$ maps to τ), the existence of the splitting implies that $c(\tau)$ is trivial at each finite place. Let $\tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ be such that ν is the chosen imbedding $E \to \overline{\mathbb{Q}}$ composed with τ (recall that E is Galois) and let (T, h, 1) be a special point of $S(K)(\mathbb{C})$ (see 3.1). Let T^{ad} be the image of T in G^{ad} and let $\mu^{ad} \in X_*(T^{ad})$ be the projection of $\mu = \mu_h \in X_*(T)$, then (because $T_{\mathbb{R}}$ is fundamental in $G_{\mathbb{R}}$) μ^{ad} satisfies the Serre condition $(\tau - 1)(\tau + 1)\mu^{ad} = 0$ for each $\tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ (ι is a non-trivial element in $Gal(\mathbb{C}/\mathbb{R})$), therefore there is a unique $\mathbb{Q}$-rational homomorphism $\chi: S^0 \to T^{ad}$ such that $\chi \circ \mu_0 = \mu^{ad}$ (μ_0 is the canonical cocharacter of S^0). The image of $c(\tau)$ in $H^1(\mathbb{Q}, G^{ad})$ by χ defines an inner twisting ${}^{v}G$ of G which is trivial at each finite place and determined by $(\tau\mu - \mu)(-1) \in T(\mathbb{C})$ at infinity, and ${}^{v}G$ is conjectured to be independent of the choise of τ and the special point. T is also a Cartan subgroup of ${}^{v}G$ and if τh: $\underline{S} \to T_{\mathbb{R}}$ is the uniquely determined $\mathbb{R}$-homomorphism for which $\mu_{\tau h} = \tau\mu$ and ${}^{v}X_{\infty}$ is the ${}^{v}G(\mathbb{R})$-conjugacy class of homomorphisms $\underline{S} \to {}^{v}G_{\mathbb{R}}$ containing τh, then ${}^{v}X_{\infty}$ is independent of the choise of τ and the special point and ${}^{\nu}G$, ${}^{\nu}X_\infty$ satisfies the conditions for G, X_∞ in 1.1. If we let ${}^{\nu}K$ be the image of K by the canonical isomorphism $G(\mathbb{A}_f) \xrightarrow{\sim} {}^{\nu}G(\mathbb{A}_f)$, the Shimura variety associated to ${}^{\nu}G$, ${}^{\nu}X_\infty$ and ${}^{\nu}K$ should be $S_\nu(K)$. If we twist the representation space V of ξ in the same way as G, we get a rational representation ${}^{\nu}\xi$ of ${}^{\nu}G$ on ${}^{\nu}V$, and the sheaf $F_{\xi,\nu}(K)$ over $S_\nu(K)(\mathbb{C})$ is ${}^{\nu}V(\mathbb{Q})\times_{\nu G(\mathbb{R}),\nu\xi}{}^{\nu}G(\mathbb{A})/{}^{\nu}K_\infty{}^{\nu}K$ (${}^{\nu}K_\infty$ is the centralizer of τh in ${}^{\nu}G(\mathbb{R})$).

By the theory of continouos cohomology we have

$$H^i(S_\nu(K)(\mathbb{C}), F_{\xi,\nu}(K))\otimes_{\mathbb{Q}}\mathbb{C} = \oplus\, H^i({}^{\nu}\overline{g}_\infty, {}^{\nu}\overline{K}_\infty, {}^{\nu}\xi\otimes\pi_\infty)\otimes\pi^{\nu K}{}_f \quad (*)$$

where the sum is taken over the irreducible representations π of ${}^{\nu}G(\mathbb{A})$ which occur (discretely) in $L^2({}^{\nu}G(\mathbb{Q})Z(\mathbb{R})Z_K\backslash{}^{\nu}G(\mathbb{A}))$ (of course only those for which the action of $Z(\mathbb{R})$ is given by the character ν^{-1} and the action of Z_K is trivial and counted with multiplicity), ${}^{\nu}\overline{g}_\infty$ is the Lie algebra of ${}^{\nu}G(\mathbb{R})/Z(\mathbb{R})$ and ${}^{\nu}\overline{K}_\infty = {}^{\nu}K_\infty/Z(\mathbb{R})$ (BW, VII, Theorem 5.2). The action of $W_{\overline{\nu(E)}}$ respects this decomposition (and is trivial on $\pi^{\nu K}{}_f$).

If $\varphi \in \Phi(G)_e$ contributes to (14), $m(\Pi_\infty) \neq 0$, this implies (since φ_∞ is essentially tempered) that Π_∞ is the L-packet of discrete series representations of $G(\mathbb{R})$ associated to one of the absolutly irreducible components of $\xi^\vee$, if this component is denoted $\xi^\vee{}_{\Pi\infty}$ (so that the representations in Π_∞ have the same infinitesimal character as $\xi^\vee{}_{\Pi\infty}$) we have $m(\Pi_\infty) = (-1)^d \cdot$ the multiplicity of $\xi^\vee{}_{\Pi\infty}$ in $\xi^\vee$ (see 3.8). A such φ belongs to $\Phi({}^{\nu}G)_e$ for each (because φ_∞ is elliptic) and contributes to (*) but only to the middle cohomology (that is, $i = d = \dim S(K)$ - BW, II, Theorem 5.3 and 5.4).

Conversely, if $\varphi \in \Phi({}^{\nu}G)_e$, it contributes at most to the middle cohomology of (*), and if it contributes, $\Pi(\varphi_\infty)$ is one of the above 1-packets (BW, III, Theorem 5.1), therefore $m(\Pi_\infty) \neq 0$ and since $\varphi \in \Phi(G)_e$, φ contributes to

(14).

The total tempered elliptic contribution to the zeta function at infinity is precisely the term $\Pi_{\varphi\in\Phi(G)e}$ ... of (14) where π_p is replaced by $\Pi^H_\infty = \Pi(\psi_\infty)$. This is an immediately consequence of the equivalence of representations of $W_{\mathbb{R}}$:

$$r^{H,i}_\nu \circ \psi_\infty = |\cdot|^{d/2} \cdot \mathrm{Ind}(W_{\mathbb{R}}, W_{\overline{\nu(E)}}, \rho'_d | \oplus H^d({}^\nu\overline{g}_\infty, {}^\nu\overline{K}_\infty, {}^\nu\xi_{\Pi\infty} \otimes \pi_\infty))$$
$$(\text{sum over } \pi \in \Pi^{i\eta}_{\nu,\infty})$$

- r_ν is defined in the same way as $r_{\wp,j}$ in 1.12, that is, chose $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that ν is the chosen imbedding composed with τ, since τ normalizes $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$, $1\times\tau \in {}^LG^0 \times \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ normalizes ${}^LG^0 \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ and if we restrict ${}^0r\circ\, \mathrm{ad}(1\times\tau)$ to ${}^LG^0\times\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ (or if E is not real, to ${}^LG^0$ and then induce to ${}^LG^0\times\mathrm{Gal}(\mathbb{C}/\mathbb{R})$) and lift to ${}^LG^0\times W_{\mathbb{R}}$, we get r_ν, $|\cdot|$ is the character $z \to z\overline{z}$ of $W_{\mathbb{C}}$ or $W_{\mathbb{R}}$. We shall use that the multiplicity of $\pi \in \Pi(\varphi)$ ($\varphi \in \Phi({}^\nu G)_e$) in $L^2({}^\nu G(\mathbb{Q})\, Z(\mathbb{R})Z_K\backslash{}^\nu G(\mathbb{A}))$ is $d_\varphi\, |\zeta_\varphi|^{-1}\, \Sigma <s, \pi>$ (sum over $s \in \zeta_\varphi$) that $r^{H,i}|{}^LH^0\times W_{\mathbb{R}} = \oplus_\nu\, r^{H,i}_\nu$ and that $<s, \Pi^{i\eta}_\infty> = <s, \Pi^{i\eta}_{\nu,\infty}>$).

[Proof of the above equivalence of representations of $W_{\mathbb{R}}$ - we use the terminology of l.12:

We assume first that ν is the chosen imbedding. Let δ_0 be the half sum of the positive roots of T_0 in G for the order making $\Lambda_0 \in X_*({}^LT_0)\otimes\mathbb{R}$ dominant, and let $\gamma_0 \in X^*(T_0)$ be the highest weight of ${}^\nu\xi_{\Pi\infty}$ w.r.t. this order. Then $\Lambda_0 = \gamma_0 + \delta_0$.

Let $G(\mathbb{R})^\bullet = T_0(\mathbb{R})G_{der}(\mathbb{R})^0 = Z(\mathbb{R})G(\mathbb{R})^0$. The representation $\pi \in \Pi_\infty$ attached to $\lambda \in \Omega_\lambda$ is obtained by inducing to $G(\mathbb{R})$ the discrete series representation of $G(\mathbb{R})^\bullet$ attached to λ. The restriction of π to $G(\mathbb{R})^\bullet$ is the direct sum of the representations of $G(\mathbb{R})^{\bullet}$ attached to $\Omega(G(\mathbb{R}), T_0(\mathbb{R}))\cdot\lambda$. For $\pi \neq \pi'$ these two sets of representations of $G(\mathbb{R})^{\bullet}$ are disjoint. The set of representations of $G(\mathbb{R})^{\bullet}$ attached to the set of characters Ω_λ has the same cardinality as Ω_μ. A one-to-one correspondance is established by letting $\mu = \omega\mu_{h0}$ corresponds to the representation attached to $\omega^{-1}\lambda_0$ ($= \lambda_0\cdot\omega$).

If $\varphi_\infty(\tau) = n\times\tau$ ($n \in \mathrm{Norm}_{LG0}({}^LT^0)$) we let $\overline{\omega} = \overline{n}\cdot{}^LT^0 \in \Omega({}^LG^0, {}^LT^0)$, and for $\mu \in \Omega_\mu$ we let $\overline{\mu} = \overline{\omega}\mu$, then $\overline{\mu} = \iota'\mu$ and $\overline{\mu} \neq \mu$ if E is real. The operator ${}^0r(n)$ - denoted by $u \to nu$ - transforms the weight space corresponding to μ to that corresponding to $\overline{\mu}$. If E is real and $\pi^{\bullet}$ is the representation of $G(\mathbb{R})^{\bullet}$ attached to $\lambda \in \Omega_\lambda$, we let $\overline{\pi}^{\bullet}$ be that attached to $\overline{\omega}\lambda$ (we note that $\overline{\omega} \in \Omega(G(\mathbb{R}), T_0(\mathbb{R}))$ (MS2, Corollary 4.3), therefore $\pi^{\bullet}$ and $\overline{\pi}^{\bullet}$ induce to the same representation of $G(\mathbb{R})$), if $\pi^{\bullet}$ corresponds to μ then $\overline{\pi}^{\bullet}$ corresponds to $\overline{\mu}$.

For $\mu \in \Omega_\mu$ let $\mathbb{C}_\mu$ be the restriction of ${}^0r\circ\varphi_\infty|W_{\mathbb{C}}$ to the weight space of ${}^0r$ corresponding to μ, and let $\mathbb{C}_\mu\oplus\mathbb{C}_{\overline{\mu}}$ be the representation of $W_{\mathbb{R}}$ given on $W_{\mathbb{C}}$ as $\mathbb{C}_\mu\oplus\mathbb{C}_{\overline{\mu}}$ and let τ acts as $\mu\oplus\overline{\mu} \to \iota(n)\,\overline{\mu}\oplus n\mu$. Then we have

$$ {}^0r\circ\varphi_\infty|W_{\mathbb{C}} \sim \oplus\mathbb{C}_\mu \text{ (sum over } \mu \in \Omega_\mu) $$

and

$$ r\circ\varphi_\infty \sim \oplus(\mathbb{C}_\mu\oplus\mathbb{C}_{\overline{\mu}}) \text{ if E is real} $$
$$ \text{(sum over } \mu \in \Omega_\mu/\sim,\ \mu' \sim \mu \Leftrightarrow \mu' = \overline{\mu}) $$
$$ \oplus(\mathbb{C}_\mu\oplus\mathbb{C}_{\overline{\mu}}) \text{ if E is not real} $$
$$ \text{(sum over } \mu \in \Omega_\mu). $$

If we induce $\mathbb{C}_\mu$ to $W_{\mathbb{R}}$, we get a representation on $\mathbb{C}_\mu\oplus\mathbb{C}_\mu$: $z \in \mathbb{C}^\times$ acts as $z\oplus\overline{z}$ and τ acts as $u\oplus u' \to (-1)^d u'\oplus u$. This representation is equivalent to $\mathbb{C}_\mu\oplus\mathbb{C}_{\overline{\mu}}$ (an equivalence is given by $\mu\oplus\mu' \to \mu\oplus n\mu'$). Therefore we have if E is not real:

$$r\circ\varphi_\infty \sim \mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, {}^0r\circ\varphi_\infty|W_{\mathbb{C}}).$$

If $\pi^\bullet$ is the representation of $G(\mathbb{R})^\bullet$ corresponding to μ, $|\cdot|^{d/2}\, \rho'_d\, H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes\pi^\bullet)|W_{\mathbb{C}}$ is equivalent to $\mathbb{C}_\mu$ and if E is real $|\cdot|^{d/2}\, \rho'_d\, H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes(\pi^\bullet\oplus\bar{\pi}^\bullet))$ is equivalent to $\mathbb{C}_\mu\oplus\mathbb{C}_{\bar{\mu}}$ (if the Cartan decomposition of $\bar{g}_\infty$ determined by ad $h_0(i)$ is $\bar{k}_\infty\oplus\bar{p}$ then $H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes\pi^\bullet) = \mathrm{Hom}_{K\infty}(\wedge^d\bar{p}_{\mathbb{C}}, \xi_{\Pi\infty}\otimes\pi^\bullet)$ and this space is one-dimensional (BW, II,Theorem 5.3) of type (p, q) with $p = d/2 - <\mu, \delta_0>$, and $q = d/2 + <\mu, \delta_0>$ and the Hodge structure is given by $z \to z^{-p'}\bar{z}^{-q'}$ with $p' = d/2 - <\mu, \Lambda_0>$ and $q' = d/2 - <\mu, \iota'\Lambda_0>$. If E is real ι^* maps $H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes\pi^\bullet)$ to $H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes\bar{\pi}^\bullet)$ and if n $\in G(\mathbb{R})$ represents $\bar{\omega} \in \Omega(G(\mathbb{R}), T_0(\mathbb{R}))$ ι^* is determined by the map on $\wedge^d\bar{p}_{\mathbb{C}}$ given by ad(n) and the map $\xi_{\Pi\infty}\otimes\pi^\bullet \to \xi_{\Pi\infty}\otimes\bar{\pi}^\bullet$ given by $(\xi_{\Pi\infty}\otimes\pi)(n)$ (π is $\pi^\bullet$ (or $\bar{\pi}^\bullet$) induced to $G(\mathbb{R})$), this operator intertwines $\xi_{\Pi\infty}\otimes\pi^\bullet$ and $(\xi_{\Pi\infty}\otimes\pi^\bullet)\circ(\mathrm{ad}(n))$.

We conclude that

$$^0r\circ\varphi_\infty|W_{\mathbb{C}} \sim |\cdot|^{d/2}\, \rho'_d\oplus H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes\pi)|W_{\mathbb{C}}$$

and if E is real

$$r\circ\varphi_\infty \sim |\cdot|^{d/2}\, \rho'_d\oplus H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes\pi)$$

(sum over $\pi \in \Pi_\infty$). By inducing in the first case for (E not real) we have in both cases

$$r\circ\varphi_\infty \sim |\cdot|^{d/2}\, \mathrm{Ind}(W_{\mathbb{R}}, W_{\bar{E}}, \rho'_d\oplus H^d(\bar{g}_\infty, \bar{K}_\infty, \xi_{\Pi\infty}\otimes\pi)).$$

It is clear from the definition of the correspondance $\Omega_\mu \leftrightarrow \{\pi^0\}$ and of $r^{H,i}$ and $\Pi^{i\eta}{}_\infty$ that this equivalence respects our decomposition when restricting to ${}^LH^0\times W_{\mathbb{R}}$.

The formulas for ν not the chosen imbedding is now an immediate consequence the fact that $(T_0)(\mathbb{R})$ is also a fundamental of Cartan subgroup of ${}^\nu G(\mathbb{R})$ and that if $\pi^\bullet$ (as a representation of $G(\mathbb{R})^\bullet$) corresponds to $\mu \in \Omega_\mu$, then $\pi^\bullet$

(as a representation of ${}^{v}G(\mathbb{R})^{\bullet}$) corresponds to $\tau\mu \in \Omega_{\tau\mu}$ if v is the chosen imbedding composed with $\tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.]

Almost all the statements in the following are conjectures - a reference is given if the conjecture is not a fabrication of mine.

Let Π be a L-packet of representations of $G(\mathbb{A})$, that is, Π is the restricted product over all places v of $\mathbb{Q}$ of L-packets Π_v of representations of $G(\mathbb{Q}_v)$, almost all Π_v are demanded to contain an unique representation which contains the trivial representation of K_v - we identify $\{\pi_v\} \in \Pi$ and $\otimes_v \pi_v$. Π is *automorphic* if some $\pi \in \Pi$ is automorphic. If some $\pi \in \Pi$ occurs (discretely) in $L^2(G(\mathbb{Q})\, Z(\mathbb{R})\backslash G(\mathbb{A}))$ then the same is true for every automorphic $\pi \in \Pi$. Π is *cuspidal* if some $\pi \in \Pi$ is cuspidal, then every automorphic $\pi \in \Pi$ is cuspidal. Π is *isobaric* if $\Pi = \Pi(\varphi)$ for some $\varphi \in \Phi(G)$, then Π is automorphic (follows from the proposition of L4 if we have proved that $\Pi(\varphi)$ is cuspidal for φ elliptic). Π is *anomalous* if it is automorphic but not isobaric. For $G = GL(n)$, Π is always singleton (Bo) and Π is isobaric if it is cuspidal (conjecture B of L5 and the conjecture (also of L5) that a tempered L-packet is of the form $\Pi(\varphi)$, in fact, a cuspida1 representation of $GL(n, \mathbb{A})$ is *per definition* isobaric in L5).

To every pair (M, Π^0) (up to conjugation by an element of $G(\mathbb{Q})$) where M is a $\mathbb{Q}$-Levy subgroup of G and Π^0 is a cuspidal L-packet of representations of $M(\mathbb{A})$, we can construct a set $\overline{\Pi}(M, \Pi^0)$ of automorphic L-packets of representations of $G(\mathbb{A})$: for each place v of $\mathbb{Q}$, the set $\{\pi \mid \exists\sigma \in \Pi^0_v\colon \pi$ is a constituent of $Ind(G(\mathbb{Q}_v), P(\mathbb{Q}_v), \sigma)$ (P some $\mathbb{Q}$-parabolic subgroup of G containing M as a Levy subgroup)$\}$ is a finite union of L-packets of representations of $G(\mathbb{Q}_v)$, Π^0_v lifted to $G(\mathbb{Q}_v)$ (via ${}^LM_v \subset {}^LG_v$ (in the following we let ${}^LG$ denote ${}^LG^0\times...$) and the principle of functoriality) is one of these L-packets (the inductive property of the (conjectural) Langlands correspondance) and this satisfies the above condition for almost all ν, we can therefore form the restricted products of all combinations of these local L-packets - every such (global) L-packet is automorphic (proved in L4). Every automorphic L-packet belongs to $\overline{\Pi}(M, \Pi^0)$ for some (M, Π^0) (proved in L4) and the sets $\overline{\Pi}(M, \Pi^0)$ are disjoint (conjecture A of L5 for G = GL(n)). Π^0 lifted to $G(\mathbb{A})$ is a L-packet in $\overline{\Pi}(M, \Pi^0)$, it is denoted by $\Pi(M, \Pi^0)$. If Π is isobaric, then $\Pi = \Pi(M, \Pi^0)$, where, if $\Pi = \Pi(\varphi)$, M is the Levy subgroup of G corresponding to the minimal relevant Levy subgroup ${}^LM$ of ${}^LG$ containing Im φ, and $\Pi^0 = \Pi(\varphi_M)$ for $\varphi_M = \varphi$ regarded as mapping into ${}^LM$ (the definition of the principle of functoriality). For G = GL(n), the isobaric L-packet are precisely those of the form $\Pi(M, \Pi^0)$ (because Π^0 is always isobaric).

A L-packet Π is *tempered* if it is automorphic and each Π_v is tempered, then Π is isobaric (L5) (the corresponding φ is tempered and conversely) and the set $\overline{\Pi}(M, \Pi^0)$ is singleton (= {Π}) (if an irreducible tempered representation of a (local) Levy subgroup is induced, the constituents should belong to the same L-packet).

If the automorphic L-packet Π is isobaric, say Π = Π(φ) for $\varphi \in \Phi(G)$, we expect that the group $\zeta_\varphi = \pi_0(S_\varphi/Z)$ and the pairing $<\,,\,>: \zeta_\varphi\times\Pi \to \mathbb{C}$ control the automorphic representations $\pi \in \Pi$: the multiplicity with which π occur in the space of automorphic forms is $d_\varphi\,|\zeta_\varphi|^{-1}\,\Sigma <s, \pi>$ (sum over $s \in \zeta_\varphi$), here d_φ is the number of (global) equivalence classes in the local equivalence class containing φ. If Π is anomalous I guess that the automorphic representations in Π are controlled by a group of the same type: we can find a φ (belonging to $\Phi(G')$ for some inner form G' of G) such that $\Pi(\varphi)$ and Π are equal at almost all places and a pairing $<\,,\,>: \zeta_\varphi \times \Pi \to \mathbb{C}$ having the above property.

According to the theory of Arthur (A1) the L-packets Π which "occur" in the regular representation of $G(\mathbb{A})$ should be parametrized by "admissible" homomorphisms $\bar\varphi$: $L_\mathbb{Q} \times SL_2(\mathbb{C}) \to {}^LG$ in the same way as the isobaric Π are parametrized by admissible homomorphisms $\varphi: L_\mathbb{Q} \to {}^LG$, however different Π can be associated to the same $\bar\varphi$, but these Π belong to the same set $\bar\Pi(M, \Pi^0)$: the φ parametrizes some of these sets. The $\varphi \in \Phi(G')$ associated to Π is in this case given by $\varphi(w) = \bar\varphi(w, \mathrm{dia}(|w|^{½}, |w|^{-½}))$ and the association $\bar\varphi \to \varphi$ is injective. If $\varphi \in \Phi(G)$, $\Pi(\varphi)$ is associated to $\bar\varphi$ (and is the isobaric L-packet (that is $\Pi(M, \Pi^0)$) in the set $\bar\Pi(M, \Pi^0)$ associated to $\bar\varphi$, for $G = GL(n)$, $\Pi(\varphi)$ is the only L-packet associated to $\bar\varphi$). In the definition of admissibility it is required that $\bar\varphi|L_\mathbb{Q}$ is essentially tempered (for $\bar\varphi|SL^2(\mathbb{C})$ trivial, $\Pi(\bar\varphi|L_\mathbb{Q})$ is the (only) L-packet associated to $\bar\varphi$). We let $\bar\Phi(G)$ denote the set (of equivalence classes) of Arthur parametres.

There is a sign character $\varepsilon_{\bar\varphi}: \zeta_{\bar\varphi} \times \Pi \to \{\pm 1\}$ and there should be a pairing $<\,,\,>': \zeta_{\bar\varphi} \times \Pi \to \mathbb{C}$ such that the multiplicity with which $\pi \in \Pi$ occurs in the regular representation is $d_{\bar\varphi}\, |\zeta_{\bar\varphi}|^{-1}\, \Sigma <s, \pi>'$ (sum over $s \in \zeta_{\bar\varphi}$). Π occurs discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$ iff $\bar\varphi$ is elliptic. We let $s_{\bar\varphi}$ denote $\bar\varphi(1 \times (-1)) \in S_{\bar\varphi}$ and its image in $\zeta_{\bar\varphi}$. $S_{\bar\varphi}$ is a subgroup of S_φ and the homomorphism $\zeta_{\bar\varphi} \to \zeta_\varphi$ is surjective (and maps $s_{\bar\varphi}$ to 1). We define a new pairing $<\,,\,>: \zeta_{\bar\varphi} \times \Pi \to \mathbb{C}$ by $<s, \pi> = ½(\varepsilon_{\bar\varphi}(s) <s, \pi>' + \varepsilon_{\bar\varphi}(ss_{\bar\varphi}) <ss_{\bar\varphi}, \pi>')$, then the multiplicity formula reads $d_{\bar\varphi}\, |\zeta_{\bar\varphi}|^{-1}\, \Sigma <s, \pi> \;=\; d_\varphi\, |\zeta_\varphi|^{-1}$

Σ <s, π> (sum over $s \in \zeta_{\bar\varphi}, \zeta_\varphi$) (< , > should factorize through $\zeta_{\bar\varphi} \to \zeta_\varphi$).

If $\bar\varphi \in \bar\Phi(G)$ and $s \in S_{\bar\varphi}$ we can (in the same way as in 1.14) construct an endoscopic datum (H, s, η) (up to isomorphism) and a $\bar\psi \in {}_G\bar\Phi(H)$ such that $\eta(\bar s) = s$ and $\eta' \circ \bar\psi \sim \bar\varphi$. This construction determines an equivalence relation ~ on $S_{\bar\varphi}$: s ~ s' the ⇔ constructed $(H, \bar s, \eta)$ and $\bar\psi$ are the same. We let $\zeta_{\bar\varphi}^* = S_{\bar\varphi}/\sim$, this set is finite and the projection $S_{\bar\varphi} \to \zeta_{\bar\varphi}$/conjugation should factorize through $S_{\bar\varphi} \to \zeta_{\bar\varphi}^*$, thus we have a projection $\zeta_{\bar\varphi}^* \to \zeta_{\bar\varphi}$/conjugation. The same construction applies to $\varphi \in \Phi(G)$, $s \in S_\varphi$.

If φ is associated to $\bar\varphi$, we have an injection $\zeta_{\bar\varphi}^* \to \zeta_\varphi^*$. If $\bar\varphi \in \bar\Phi(G)_e$ (e = elliptic), $\zeta_{\bar\varphi} = S_{\bar\varphi}/Z$, and this group and ζ_φ are abelian. The image of $\zeta_{\bar\varphi}^* = \zeta_{\bar\varphi}$ in ζ_φ^* is denoted $(\zeta_\varphi^*)_f$.

Proposition 11.3.2 of K3 (see 1.14) should remain true for $\bar\varphi \in \bar\Phi(G)_e$, and also the (conjectural) considerations at the end of that paper: if $\bar\varphi \in \bar\Phi(G)_e$, its contribution to the trace $\Sigma_{\pi\sim\bar\varphi} \Sigma_{\pi\in\Pi}\, m_\pi \operatorname{tr} \pi(\varphi)$ (φ a function on $G(\mathbb{A})$) can be stabelized as:

$$\Sigma_{(H,s,\eta)\in\mathscr{E}}\, \iota(G, H)\, \Sigma_{\Pi H}\, \Sigma_{\pi\in\Pi H}\, n_\pi \operatorname{tr} \pi(\varphi^H),$$

here Π^H runs over the automorphic L-packets of representations of $H(\mathbb{A})$ which lift to some Π associated to $\bar\varphi$, φ^H is a function on $H(\mathbb{A})$ connected with φ (see 3.7, in the formula there we must replace $\Phi(G)_{temp}$ by $\bar\Phi(G)$, $\Phi(H)_{temp}$ by $\bar\Phi(H)$, <1, π> by $\langle s_{\bar\psi}, \pi\rangle'$, <η(s), π> by $\langle \eta(s)s_{\bar\varphi}, \pi\rangle'$ and the summation must be taken over all Π^H resp. Π associated to $\bar\psi$ resp. $\bar\varphi$) and $n_\pi = d_{\bar\psi}\, |\zeta_{\bar\psi}|^{-1}\, \varepsilon_{\bar\psi} \langle s_{\bar\psi}, \pi\rangle'$ is the stable multiplicity of π.

Now I can state the complete form of the expression for the zeta function in terms of L-functions:

$$\Pi_\Pi \Pi_{s\in(\zeta\varphi^*)f} (\Pi_{\varepsilon\in\{\pm1\}} (\Pi_{i\in H(s)} L(s - d/2, \psi_M, r_\varepsilon^{H,i})^b)^a) \ (**)$$
$$a = \varepsilon\, m(\Pi^0{}_\infty)\, d_\varphi\, |(\zeta_\varphi^*)_f|^{-1}$$
$$b = \Sigma_{\pi f\in\Pi f} <s, \Pi^{i\eta,\varepsilon}{}_\infty\otimes\pi_f> \mathrm{tr}\, \pi_f(\varphi)$$

(here and in some of the following formulas we should strictly speaking change the sign in $m(\Pi^0{}_\infty)$, since we have defined the zeta function as the inverse product of the local zeta functions). In the formula Π runs over the L-packets of representations of $G(\mathbb{A})$ occuring (discretely) in $L^2(G(\mathbb{Q})\, Z(\mathbb{R})\backslash G(\mathbb{A}))$ (and for which $Z(\mathbb{R})$ and Z_K act as usual). φ ($\in \Phi(G')$) is associated to Π as above. Let $\overline{\varphi} \in \overline{\Phi}(G)_e$ be an Arthur parameter of Π, we can assume that $\overline{\varphi}_\infty(W_\mathbb{C}) \subset {}^LT^0\otimes W_\mathbb{C}$. The centralizer ${}^LM^0$ of $\overline{\varphi}_\infty(W_\mathbb{C})$ in ${}^LG^0$ is a Levy subgroup (containing ${}^LT^0$) and if $\overline{\varphi}_\infty(z) = z^\Lambda \overline{z}^{\iota\Lambda} \times z$ ($\Lambda \in X_*(Z_{LM0})\otimes\mathbb{R}$), Λ determines a parabolic subgroup ${}^LP^0$ of ${}^LG^0$ with ${}^LM^0$ as Levy subgroup, $\overline{\varphi}_\infty(\tau)$ determines an action of $Gal(\mathbb{C}/\mathbb{R})$ on ${}^LM^0$. If $\varphi'_M \in \Phi(M)$ parametrizes the "trivial" discrete series representation of $M(\mathbb{R})$, then $\varphi^0{}_\infty: W_\mathbb{R} \to^{\varphi'M} {}^LM^0\times W_\mathbb{R} \to^a {}^LG^0\times W_\mathbb{R}$ ($a = id\times \overline{\varphi}_\infty|W_\mathbb{R}$) belongs to $\Phi(G'_\infty)$ (G'_∞ quasi-split form of G_∞). We can restrict our attention to those $\overline{\varphi}$ for which $\overline{\varphi}_\infty$ and $\varphi_\infty{}^0$ are elliptic (and $\overline{\varphi}(1, \{1\ 1\ /\ 0\ 1\})$ is regular uni-potent in ${}^LM^0$), and we let $\Pi^0{}_\infty = \Pi(\varphi^0{}_\infty)$. To φ and $s \in (\zeta_\varphi^*)_f$ we construct a (H, s, η) and a $\psi \in \Phi(H)$ as above. Define $\mu_{h0} \in X^*({}^LT^0)$ as in 1.9 ((H, s, η) need not be elliptic at infinity, but we can restrict our attention to those φ for which T_0 can be chosen elliptic at infinity) and define $\mu_0 \in X^*({}^LT^0)$ (from $\varphi^0{}_\infty$) as in 1.12, then $\eta = (\mu - \mu_{h0})(s)$ and $\mathscr{H}(s) = \{(\mu - \mu_0)(s) \mid \mu \in \Omega_\mu\}$ ($\subset$ roots of unity). A $\pi_\infty \in \Pi_\infty$ (for which $<s, \pi_\infty>' \neq 0$ for some $s \in S_{\overline{\varphi}\infty}$) is constructed from a Levy subgroup M of $G_\mathbb{R}$ and a parabolic subgroup P of $G_\mathbb{C}$ containing M as Levy subgroup. We can choose a fundamental Cartan subgroup T of $G_\mathbb{R}$ contained in M and a

$h \in X_\infty$ factoring through T. The L-group of M is ${}^LM^0\times$ Gal($\mathbb{C}/\mathbb{R}$), in this construction we have chosen an isomorphism $X_*(T) \leftrightarrow X^*({}^LT^0)$ "transforming" P to ${}^LP^0$. To π_∞ we associate the element $i' = (\mu - \mu_0)(s) \in \mathscr{H}_h(s) = \{(\mu - \mu_{h0})(s) \mid \mu \in \Omega_\mu\}$ (this is well defined) and this association determines a disjoint family of subsets $\Pi^{i'}{}_\infty \subset \Pi_\infty$ ($\Pi^{i'}{}_\infty$ can be empty), we let $\Pi^{i',\varepsilon}{}_\infty = \{\pi_\infty \in \Pi^{i'}{}_\infty \mid \mu_h(s_{\bar\varphi}) = \varepsilon\}$. We have a bijection $\mathscr{H}(s) \leftrightarrow \mathscr{H}_h(s)$ given by $i \rightarrow i' = i\eta$. If ${}^LM$ resp. ${}^LM^H$ is the minimal Levy subgroup of ${}^LG$ resp. ${}^LH$ containing Im φ resp. Im ψ, then $s_{\bar\varphi} \in Z_{LM}$ resp. Z_{LMH} and $s_{\bar\varphi}$ determines a $\pm$-decomposition of $r|L_M$ resp. $r^{H,i}|{}^LM^H$.

The proof is an immediate generalizasion of step (l0)-(14) in section 2: In (l0) we shall replace $\Phi(H)_e$ by $\bar\Phi(H)_e$, $\Pi(\psi)$ by Σ (sum over $\Pi^H \sim \bar\psi$) and <1, π> by $\varepsilon_{\bar\psi}(s_{\bar\psi})$ $<s_{\bar\psi}, \pi>'$. $m(\Pi_\infty)$ must be replaced by $\Sigma\,\Sigma <s_{\bar\psi}, \pi>'$ tr $\pi(f^G{}_\xi)$ (sum over $\Pi_\infty \sim \bar\varphi_\infty$, $\pi \in \Pi^0{}_\infty$) and this should be equal to $<s_{\bar\varphi\infty}, \pi_\infty>'$ $\mu_h(s_{\bar\varphi\infty})$ $m(\Pi_\infty{}^0)$, where π_∞ (arbitrary) is associated to $\bar\varphi_\infty$. A similar change of $m(\Pi^H{}_\infty)$. We note the generalizations of 3.6 and 3.7. In (14) we shall incorporate $\varepsilon_{\bar\varphi}(ss_{\bar\varphi})$ and Σ (sum over $\Pi \sim \bar\varphi$) and replace <s, π> by $<ss_{\bar\varphi}, \pi>'$ (we use that $\varepsilon_{\bar\psi}(s_{\bar\psi}) = \varepsilon_{\bar\varphi}(ss_{\bar\varphi})$). We have a bijection $\{i \in \mathscr{H}(s) \mid r^{H,i}{}_\varepsilon \neq 0\} \rightarrow \{i' \in \mathscr{H}(ss_{\bar\varphi}) \mid r^{H',i'}{}_\varepsilon \neq 0\}$ given by $i \rightarrow i' = \varepsilon\mu_0(s_{\bar\varphi})i$. Now the formula follows from the fact that $r^{H,i}{}_\varepsilon \circ \psi_M = r^{H',i'}{}_\varepsilon \circ \psi_M'$ and

$$\tfrac{1}{2}(\varepsilon_{\bar\varphi}(ss_{\bar\varphi}) <s, \Pi^{i\eta}{}_\infty>' \Sigma <ss_{\bar\varphi}, \pi_f>' \operatorname{tr} \pi_f(\varphi)$$
$$+ (\varepsilon_{\bar\varphi}(s) <ss_{\bar\varphi}, \Pi^{i'\eta'}{}_\infty>' \Sigma <s, \pi_f>' \operatorname{tr} \pi_f(\varphi)$$
$$= <s_{\bar\varphi}, \pi_\infty>' \Sigma <s^-, \Pi^{i\eta,\varepsilon}{}_\infty \otimes \pi_f>' \operatorname{tr} \pi_f(\varphi)$$
$$\text{(sum over } \pi_f \in \Pi_f),$$

where π_∞ (arbitrary) $\in \Pi^{i\eta,\varepsilon}{}_\infty$. Of course this "proof" works only locally at primes p satisfying our conditions in this paper.

Now we will compare this formula with a formula for the zeta function obtained from a decomposition of the étal cohomology parametrized by representations analogous to that of the rational cohomology used in our proof of (14) at the infinete place.

$G(\mathbb{A}_f)$ and so the Hecke algebra at $\mathcal{H}(G(\mathbb{A}_f), K)$ (with coefficients in $\mathbb{Q}$ resp. $\mathbb{Q}_\ell$) acts on $H^i(S(K)(\mathbb{C}), F_\xi(K))$ and $H^i_{ét}(S(K)(\mathbb{C}), \zeta_\xi(K)_{\mathbb{Q}\ell})$ (if $g \in G(\mathbb{A}_f)$ and $K' = K \cap gKg^{-1}$ we have two morphisms $S(K') \to S(K)$ (defined over E) a) by right multiplication by g and b) by inclusion, these induce maps on cohomology:

$$H^i(S(K)(\mathbb{C}), F_\xi(K)) \to H^i(S(K')(\mathbb{C}), F_\xi(K')) \to H^i(S(K)(\mathbb{C}), F_\xi(K))$$

and

$$H^i_{ét}(S(K)(\mathbb{C}), \zeta_\xi(K)_{\mathbb{Q}\ell}) \to H^i_{ét}(S(K')(\mathbb{C}), \zeta_\xi(K')_{\mathbb{Q}\ell}) \to H^i_{ét}(S(K)(\mathbb{C}), \zeta_\xi(K)_{\mathbb{Q}\ell}),$$

the left maps because the inverse image by a) of $F_\xi(K)$ resp. $\zeta_\xi(K)$ is $F_\xi(K')$ resp. $\zeta_\xi(K')$, the right maps because we have a map from the direct image by b) of $\zeta_\xi(K')$ resp. $\zeta_\xi(K)$ to $F_\xi(K)$ resp. $\zeta_\xi(K)$).

The actions of $\mathcal{H}(G(\mathbb{A}_f), K)_{\mathbb{Q}\ell}$ and $Gal(\overline{E}/E)$ on $H^i_{ét}(S(K)(\mathbb{C}), \zeta_\xi(K)_{\mathbb{Q}\ell})$ commute and lead to a decomposition

$$H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell}) \otimes \overline{\mathbb{Q}}_\ell = \oplus_\pi X^i(\pi_\infty) \otimes W(\pi_f)$$

(π as before), $X^i(\pi_\infty)$ is a $Gal(\overline{E}/E)$-module and depends on π, $W(\pi_f)$ is an irreducible $\mathcal{H}(G(\mathbb{A}_f), K)_{\mathbb{Q}\ell}$-module. If we choose an imbedding $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and tensorize both sides we get the former decomposition of $H^i(S(K), F_\xi(K)) \otimes_{\mathbb{Q}} \mathbb{C}$ ($X^i(\pi_\infty) \otimes_{\overline{\mathbb{Q}}\ell} \mathbb{C} = H^i(\overline{g}_\infty, \overline{K}_\infty, \xi \otimes \pi_\infty)$ and $W(\pi_f) \otimes_{\overline{\mathbb{Q}}\ell} \mathbb{C} = \pi^K{}_f$), in fact, we have obtained the decomposition of $H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell}) \otimes \overline{\mathbb{Q}}_\ell$ by first decomposing into irreducible $\mathcal{H}$(G

$(\mathbb{A}_f)$, K$)_{\mathbb{Q}\ell}$-modules and then comparing with the decomposition of $H^i(S(K), F_\xi(K))\otimes_{\mathbb{Q}}\mathbb{C}$.

If the L-packet Π contributes to the above sum and $\pi_\infty \in \Pi_\infty$, then if $\pi = \pi_\infty\otimes\pi_f$ contributes to the sum for some $\pi_f \in \Pi_f$, we expect that the $\mathrm{Gal}(\bar{E}/E)$-module $X^i(\pi_\infty)$ is independent of the choise of π_f in Π_f, hence we can define the Gal $(\bar{E}/E)$-module $X^i(\Pi_\infty) = \oplus_{\pi\infty\in\Pi\infty} X^i(\pi_\infty)$ (depending on Π).

For every finite place ν of E we thus have (for $\ell \neq \nu$) a λ-adic representation $\rho'^i_\nu(\Pi_\infty)$ of $W_{E\nu}$ (via $W_{E\nu} \rightarrow \mathrm{Gal}(\bar{E}_\nu/E_\nu)$) on $X^i(\Pi_\infty)$ ($X^i(\Pi_\infty)$ should be replaced by a vector space over some finite extension of $\mathbb{Q}_\ell$), and for every infinite place ν of E we have the former (complex) representation $\rho'^i_\nu(\Pi_\infty)$ of $W_{E\nu}$ on $X^i(\Pi_\infty) \otimes_{\bar{\mathbb{Q}}\ell}\mathbb{C}$. By inducing we have a representation $\rho^i_\nu(\Pi_\infty)$ of $W_{\mathbb{Q}\bar{\nu}}$ ($\bar{\nu}$ the place of $\mathbb{Q}$ divided by ν).

This decomposition of the cohomology imply:

$$Z(s, S(K), \xi) = \Pi_\nu\, \Pi_\pi\, \Pi_j\, (\Pi_\nu\, L(s, \rho^j_\nu(\pi_\infty))^{\dim \pi Kf})^{(-1)\text{^}j}$$
$$= \Pi_\Pi\, \Pi_s\, \Pi_j\, \Pi_{i,\varepsilon}\, (\Pi_\nu\, L(s, \rho^j_\nu(\Pi^{i,\varepsilon}{}_\infty))^a)^{(-1)\text{^}j}$$
$$(s \in (\zeta_\varphi{}^*)_f,\ i \in \mathscr{H}_{h0}(s),\ \varepsilon = \pm 1),$$

where $a = d_\varphi\, |(\zeta_\varphi{}^*)_f|^{-1}\, \Sigma_{\pi f\,\in\,\Pi f} \langle s, \Pi^{i,\varepsilon}{}_\infty\otimes\pi_f\rangle\ \mathrm{tr}\ \pi_f(\varphi))$, and thus we should have (in order to deduce this we must incorporate a dependence of the Hecke algebra in the zeta fımction, see BL):

$$\Pi_{\pi\sim\bar{\varphi}}\, \Pi_j\, (\Pi_\nu\, L(s, \rho^j_\nu(\Pi_\infty)))$$
$$= L(s - d/2, \varphi_M, r_\varepsilon)^{|m(\Pi 0\infty)|}\ ((-1)^{d+j} = \varepsilon),$$

if $\langle 1, \pi\rangle \neq 0$ for some of the π associated to $\bar{\varphi}$ for which $\mu_h(s_{\bar{\varphi}}) = \varepsilon$ $(\pi_\infty \rightarrow \mu_h)$, here both sides should decompose in accordance with i, that is, the subsets $\Pi^{i,\eta}{}_\infty$ of Π_∞ and the constituents $r^{H,i}$ of r restricted to ${}^LH$ - we expect that if π_∞ contributes to the cohomology of degree j, then $(-1)^{d+j} =$

$\mu_h(s_{\bar\varphi})$.

The λ-adic representation

$$\oplus_{\pi\sim\bar\varphi} \oplus \rho^j(\Pi_\infty) \ ((-1)^{d+j} = \varepsilon)$$

of $W_{\mathbb{Q}}$ (via $W_E \to Gal(\bar{E}/E)$ and inducing) should thus corresponds (locally) to the complex representation

$$|m(\Pi_\infty)|\cdot|\cdot|^{-d/2}\ r_\varepsilon\circ\varphi_M$$

of $L_{\mathbb{Q}}$ (for this correspondance see Ta - at an infinite place $\rho^j(\Pi_\infty)$ must be defined as earlier and is actually complex - strictly speaking a λ-adic representation must be replaced by its Φ-semi-simplificaton).

If $\bar\varphi \in \Phi(G)_e$ is such that an associated L-packet Π contributes to $H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell})\otimes\bar{\mathbb{Q}}_\ell$ for some X_∞ ($\subset$ {$\underline{S} \to G_{\mathbb{R}}$}), K and ξ, then the (λ-adic) representation $\oplus_{\pi\sim\bar\varphi} \oplus_j \rho^j(\Pi_\infty)$ $((-1)^{d+j} = \varepsilon)$ of $W_{\mathbb{C}}$ should be the $|m(\Pi^0_\infty)|$-fold of a representation $\rho_{\bar\varphi,W\infty,\varepsilon}$ which should depend only on $\bar\varphi$, X_∞ and ε, but which should be independent of ξ and K, and this should correspond to the (complex) representation $|\cdot|^{-d/2}\ r_\varepsilon\circ\varphi_M$ of $L_{\mathbb{Q}}$ (in particular dim $\rho_{\bar\varphi,W\infty,\varepsilon}$ = dim r_ε) - we expect that $m(\Pi^0_\infty) = (-1)^d$ · the multiplicity of the absolutely irreducible constituent of $^\vee\xi$ having the same infinitesimal (and central) character as Π^0_∞. Otherwise stating: the (isobaric) representation of GL(n, $\mathbb{A}$) (n = dim $\rho_{\bar\varphi,W\infty,\varepsilon}$) corresponding to $\rho_{\bar\varphi,W\infty,\varepsilon}$ (by the Langlands correspondance) should be $|\det|^{-d/2}$ · the representation of GL(n, $\mathbb{A}$) (n = dim r_ε) obtained by lifting the (cuspidal) L-packet $\Pi^0 = \Pi(\varphi_{M'})$ of representations of M'($\mathbb{A}$) via r_ε: $^LM' \to GL(n, \mathbb{C})$, here $^LM'$ is the minimal (relevant w.r.t. G') Levy subgroup of ^LG containing Im φ and M' is the Levy subgroup of G' corresponding to $^LM'$ (this is proved in Ll for G = GL(2) and π cuspidal, but only locally for some types of π_p - for this G the Shimura variety is not compact so a generalization of our theory is necessarily, see below, see also BL, HLR and Ra).

The point is now that the dependence of this representation of GL(n, $\mathbb{A}$) on the Shimura variety, that is on X_∞, should be reflexed only in r (which is constructed from X_∞), so that $\varphi \in \Phi(G')$ should be independent of S(K) and in fact should be the φ which we earlier have associated to Π.

Since a L-function L(s, φ, r) is known to converge absolutly for Re s sufficiently large, to extend meromorphic and to satisfy the functional equation $L(s, \varphi', r) = \varepsilon(s, \varphi, r)$ $L(1 - s, {}^{\vee}\varphi, r)$ (${}^{\vee}\varphi$ is the contragredient of φ, for the definition of $\varepsilon(s, \varphi, r)$ and for a proof see Ta), the zeta function (which we have regarded as a formal power series) should converge absolutly for Re s sufficiently large, extend meromorphic and satisfy a functional equation, in fact, this functional equation seems to have the expected form $Z(s, M) = \varepsilon(s, M)\, Z(1 - s, {}^{\vee}M)$ (M is a motive over an algebraic numberfield and ${}^{\wedge}M$ is the dual motive, see Ta and D2):

$$Z(s, S(K), \xi) = \varepsilon(s, S(K), \xi)\, Z(1 - s, S(K), {}^{\vee}\xi)$$

$(M(d) = M \otimes T^{\otimes d}$ - the Tate object, thus $Z(s, M(d)) = Z(s + d, M))$. If $M(S(K), \xi)$ is the motive associated to $(S(K), \xi)$, the motive which we here associate to $(S(K), \xi)$ is $\oplus_i (-1)^{i+1} M^i(S(K), \xi)$. We should have ${}^{\vee}M^i(S(K), \xi) = M^{2d-i}(S(K)(d), {}^{\vee}\xi^i) = M^i(S(K)(i), {}^{\vee}\xi)$, and so the homogeneous functional equation

$$Z^i(s, S(K), \xi) = \varepsilon^i(s, S(K), \xi)\, Z^i(i + 1 - s, S(K), {}^{\vee}\xi).$$

The functional equation follows from the fact that (the global) $\varepsilon(s, V)$ is additive and that we (by the above) have

$$\Pi_{\Pi\sim\bar{\omega}} \Pi_i \, \varepsilon(s, \rho^i(\Pi_\infty))^{(-1)^i} \; ((-1)^{d+1} = \varepsilon)$$
$$= \varepsilon(s - d/2, \varphi_M, r_\varepsilon)^{\varepsilon\, m(\Pi 0\infty)}.$$

If S(K) is not proper (that is, G_{ad} is not anisotropic over $\mathbb{Q}$), we can still easy define a zeta function. But a definition which is appropriate for an expression of the zeta function in terms of L-functions require some preliminary work.

If S (K) has "good" reduction at $\wp$, that is, if $S_\wp(K)$ is defined and smooth, we have (by the Lefschetz fixed point formula)

$$\exp \Sigma_{j=1}^{\infty} |\omega_\wp|^{js}/j \; |S_\wp(K)(\kappa^j)|$$
$$= \Pi_{i=1}^{2d} \det(1 - |\omega_\wp|^s \, \Phi_\wp | H^i(S(K), \mathbb{Q}_\ell))^{(-1)^{(i+1)}}.$$

The left hand side is clearly a $\mathbb{Q}$-rational function of $|\omega_\wp|^s$, but if $S_\wp(K)$ is not proper we can not any more prove that the individual factors on the right have coefficients in $\mathbb{Q}$. This fact is hovever in reality inessential for us, for other reasons we have to choose another cohomology. In contrast to the compact case, the eigenvalues α of the Frobenius action on the cohomology (being algebraic since the ℓ-adic polynomial has algebraic coefficients) need no rnore be "pure", that is, satisfy $\log_p |\nu(\alpha)|^2 \in \mathbb{Z}$ for every infinite place ν of the solution field - this defect already appears for GL(2).

It seems as if the cohomology used to define the zeta function ought to satisfy this purity condition. Also we must demand that it have an appropriate decomposition parametrized by representations like that of the usual (ℓ-adic) cohomology for S(K) proper. The existence of a such cohomology would for instance allow us to prove the Ramanujan-Petersson conjecture for a L-packet Π occuring discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$: if Π_∞ is discrete (and almost all Π_p have a Whittaker model), then almost all Π_p are (essentially) tempered.

It is natural to choose a suitable compactification $\overline{S(K)}$ of S(K) and extend $\zeta_\xi(K)_{\mathbb{Q}\ell}$ to $\overline{S(K)}_{\overline{E}}$, and to study the image of the restriction map $H^i_{ét}(\overline{S(K)}, \zeta_\xi(K)_{\mathbb{Q}\ell}) \to H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell})$, or the image of the map $H^i_{ét,c}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell}) \to H^i_{ét}(S(K), \zeta_\xi(K)_{\mathbb{Q}\ell})$ (c = compact support). The first cohomology is clearly $\mathbb{Q}$-rational and pure, the second is pure but the $\mathbb{Q}$-rationallity is unknown. The Hecke algebra $H(G(\mathbb{A}_f), K)$ acts semi-simply on both cohomology spaces, they therefore possess a decomposition into irreducible $H_{\mathbb{Q}\ell}$-modules, but this decomposition need not come from a decomposition parametrized by representations.

It seems as if the *intersection cohomology* $IH^i(\overline{S(K)}, \zeta_\xi(K)_{\mathbb{Q}\ell})$ (references in BL and HLR) is the adequate cohomology for the definition of the zeta function: the purity seems present and is proved for S(K) proper, and it seems to have the correct decomposition property: we have $IH^i(\overline{S(K)}, \zeta_\xi(K)_{\mathbb{Q}\ell})\otimes_{\mathbb{Q}\ell}\mathbb{C} = IH^i(\overline{S(K)}, F_\xi(K))\otimes_{\mathbb{Q}}\mathbb{C}$, and the last space seems to be isomorphic to the L^2-cohomology space $H^i_{(2)}(S(K), F_\xi(K)_{\mathbb{C}})$ (the conjecture of Zucker), but the L^2-cohomology is isomorphic to the $\mathfrak{g}$-$\mathfrak{k}$-cohomology, and this seems also in the non-compact case to possess the decomposition parametrized by representations occuring discretely in $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$.

In Ll, BL and HLR the cases of Hilbert-Blumenthal varieties are treated ($G = Res_{F/\mathbb{Q}}GL(2)$, F a real numberfield). To define the intersection cohomology we let $\overline{S(K)}$ be the Satake-Baily-Borel compactification. This is *not* smooth. S(K) and $\overline{S(K)}$ are defined over $\mathbb{Q}$ and the frontier $S(K)_\infty = \overline{S(K)}\backslash S(K)$ is finite. If $K = K_n$, S(K) is defined over spec ($\mathbb{Z}[1/n]$) and there is an open subset W of spec ($\mathbb{Z}[1/n]$) such that S(K) restricted to W has a compactification which after base-change by spec($\mathbb{Q}$) → W becomes $\overline{S(K)}$, we can also construct a *smooth* compactification of S(K) over spec($\mathbb{Z}[1/n]$) which over W is a resolution of singularities of $\overline{S(K)}$.

The expression (**)for the zeta function in terms of L-functions should hold also in the non-compact case. In the proof a non-elliptic part of the trace comes into play, that is, a part coming from other parabolic subgroups of G than G. This part is the contribution to the sum (l) from the frontier $S_{\wp}(K)_\infty$. For the above Shimura varieties this contribution is only non-zero for F = $\mathbb{Q}$, the case studied in Ll.

All the existing proofs of special (multidimensional) cases of formula (**) - where S(K) thus may be noncompact and where the reduction at p may be bad - have a look like our proof in this paper. It is always assumed that $S_{\wp}(K)$ exists for $\wp|p$ and K_p is maxiınal compact. A such proof will however in this generality, strictly speaking, leads to an expression for the *semi-simple* zeta fiınction in terms of *semi-simple* L-functions (precisely: $L^{ss}(s - d/2, \overline{\psi_{\overline{M}}}, r^{H,i}{}_{\varepsilon})$ - ψ_M must be replaced by $\overline{\psi_M}$).

The generalization of our proof can be outlined in the following way. We have a diagram:

$$\begin{array}{ccccc} \overline{S}_{\wp}\overline{(K)}_{\overline{E}\wp} & \xrightarrow{j} & \overline{S}_{\wp}\overline{(K)} & \xleftarrow{i} & \overline{S}_{\wp}\overline{(K)}_{\kappa} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{spec}(\overline{E}_{\wp}) & \rightarrow & \mathrm{spec}(O_{E\wp}) & \leftarrow & \mathrm{spec}(\kappa). \end{array}$$

Let $IC^{\bullet}(\overline{S}_{\wp}\overline{(K)}_{\overline{E}\wp}, \zeta_\xi(K)_{\mathbb{Q}\ell})$ be the cochain-complex used to define the intersection cohomology, and let, for $x \in \overline{S}_{\wp}\overline{(K)}(\kappa^j)$, $Tr_{x,j}$ be the *alternating* trace of the action of the Frobenius over κ^j on the inertia invariants in the sheaves i*H $(j_*IC^{\bullet}\ \overline{S}_{\wp}\overline{(K)}_{\overline{E}\wp}, \zeta_\xi(K)_{\mathbb{Q}\ell}))$ on $\overline{S}_{\wp}\overline{(K)}_\kappa$ at the point x (the sheaves of *vanishing cycles* - Gal($\overline{E}_{\wp}$/E) acts on these sheaves - it is conjectured that the inertia group acts through a finite factor group). In formula (1) we must replace tr $(\Phi_{\wp}{}^{j})_{x}$ by $Tr_{x,j}$ and sum over $\overline{S}_{\wp}\overline{(K)}(\kappa^{j})$. If we ignore the contribution from the frontier $S(K)_{\infty}$ to the zeta function (or assume that it is zero, that is, $Tr_{x,j} = 0$ for $x \in S_{\wp}(K)_{\infty}(\kappa^{j})$, cf. the above remark), we can in formula (2) be con-tent with replacing $|(I_{\varphi})_{\varepsilon}\backslash(Y^{j}{}_{p}\times Y^{p})|$ by $\Sigma\, Tr^{0}{}_{x,j}$ (sum over $x \in$ $A(\varphi, \varepsilon)(\kappa^{j})$), here Tr^{0} is Tr for ξ trivial and A is defined on p. ... $f^{\sim}{}_{\wp,n}$ in formula (3) have to be defined in terms of $Tr^{0}{}_{x,j}$ (see Ra). In order to get (12) we shall use that tr $\pi_{p}(f^{H}{}_{\wp,j}) = (1/j)\ |\omega^{j}|^{-d/2}\ \Sigma_{i\in\mathscr{H}}\, i\ \cdot$ [the *semi-simple* trace of the action of the j-th power of a Frobenius on the space of the ℓ-adic representation associated to the representation $^{\vee}r^{H,i}{}_{\wp,j}\circ\overline{\psi}_{p}: L_{\mathbb{Q}p}\times SL_{2}(\mathbb{C}) \rightarrow GL(V^{i}{}_{r})$].

If we assume that "the monodrony filtration of $IH^{\bullet}$ $(\overline{S}_{\wp}\overline{(K)}_{\overline{E}\wp}, \mathbb{Q}_{\ell})$ is pure" (a conjecture of Deligne, see Ra), then the proved expression for the semi-simple zeta function in terms of semi-sinple L-functions should imply our wanted formula (**).

Appendix

Definition of $\mathcal{W}$ and $\mathcal{D}$

Let v be a place (of $\mathbb{Q}$) and let K be a finite Galois extension of $\mathbb{Q}_v$, then we have an exact sequence

$$K^\times \to W_{K/\mathbb{Q}v} \to \mathrm{Gal}(K/\mathbb{Q}_v),$$

defined by a splitting $d_\sigma \in W_{K/\mathbb{Q}v}$ ($\in \mathrm{Gal}(K/\mathbb{Q}_v)$), where $d_\delta d_\sigma d_{\delta\sigma}^{-1} = d_{\delta,\sigma}$ - a 2-cocycle in the fundamental class of K/ $\mathbb{Q}_v$ - $d_\sigma k d_\sigma^{-1} = \sigma(k)$ for $k \in K^\times$. The sequence is determined up to an isomorphism which in turn is determined up to conjugation by an element of $K^\times$.

If we choose an algebraic closure $\overline{\mathbb{Q}}_v$ of $\mathbb{Q}_v$ containing K, we have, by forward and backward transform, a gerb

$$G_m(\overline{\mathbb{Q}}_v) \to \mathcal{D}^K \to \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v).$$

For $K \subset K'$ ($\subset \overline{\mathbb{Q}}_v$) we have a natural homomorphism $\mathcal{D}^{K'} \to \mathcal{D}^K$ (determined up to conjugation by an element of $G_m(\overline{\mathbb{Q}}_v)$) given by $x \to x^{[K':K]}$ (on the kernel) and $d'_\sigma \to c_\sigma d_\sigma$ if $(d'_{\delta,\sigma})^{[K':K]}/d_{\delta,\sigma} = c_\delta \delta(c_\sigma) c_{\delta\sigma}^{-1}$, and therefore we have a limit $\mathcal{D}^v = {}_{\leftarrow K}\lim \mathcal{D}^K$. Of course $\mathcal{D}^\infty = G_m(\mathbb{C}) \to W_\mathbb{R} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Definition of $\mathscr{L}$ and $\zeta: \mathcal{W}_\infty \to \mathscr{L}$, $\zeta_p: \mathcal{D}_p \to \mathscr{L}$ and $\zeta_\ell: G_\ell \to \mathscr{L}$

Let p be a prime number. We choose algebrac closures $\mathbb{C}$ of $\mathbb{R}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we choose imbeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let L ($\subset \overline{\mathbb{Q}}$) be a finite Galois extension of $\mathbb{Q}$, let $\tilde{\nu}$ be the place of L over ∞ defined by $L \subset \overline{\mathbb{Q}} \to \mathbb{C}$ and let $\wp$ be the place of L over p defined by $L \subset \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

Let $m \in \mathbb{N}$ and $q = p^m$. The set

$Y(L, m) = \{\pi \in L^* \mid$

for each place $\bar{\nu}$ of L over ∞ is $|\Pi_\sigma \sigma\pi|^{[L_{\bar{\nu}}:\mathbb{R}]} = q^{a[L:\mathbb{Q}]}$

for some $a \in \mathbb{Z}$ ($\sigma \in \mathrm{Gal}(L/\mathbb{Q})$)

for each place $\bar{\nu}$ of L over p is $|\Pi_\sigma \sigma\pi| = q^b$

for some $b \in \mathbb{Z}$ ($\sigma \in \mathrm{Gal}(L_{\bar{\nu}}/\mathbb{Q}_p)$)

for each place $\bar{\nu}$ of L over $\ell \neq p$ is π an unit$\}$

is a subgroup of $L^\times$ and $Y^*(L, m) = Y(L, m)/\{\text{units in } Y(L, m)\}$ is a finitely generated free group on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts. Let Q(L, m) be the corresponding $\mathbb{Q}$-torus (that is $X^*(Q(L, m)) = Y^*(L, m)$) and let $\nu_\infty, \nu_p \in X_*(Q(L, m))$ be defined by

$\langle\nu_\infty, \chi_\pi\rangle$ = the a in the condition for $\bar{\nu} = \tilde{\nu}$

$\langle\nu_p, \chi_\pi\rangle$ = the b in the condition for $\bar{\nu} = \wp$,

for any $\pi \in Y(L, m)$ - here χ_π is the character of Q(L, m) associated to π.

We choose imbeddings of exact sequences

$$\begin{array}{ccccc} L^{\times}_{\nu\sim} & \to & W_{L\nu\sim/\mathbb{R}} & \to & \mathrm{Gal}(L_{\nu\sim}/\mathbb{R}) \quad (\infty) \\ \downarrow & & \downarrow & & \downarrow \\ C_L & \to & W_{L/\mathbb{Q}} & \to & \mathrm{Gal}(L/\mathbb{Q}) \quad (\mathbb{Q}) \end{array}$$

and

$$\begin{array}{ccccc} L^{\times}_{\wp} & \to & W_{L\wp/\mathbb{Q}p} & \to & \mathrm{Gal}(L_{\wp}/\mathbb{Q}_p) \ (p) \\ \downarrow & & \downarrow & & \downarrow \\ C_L & \to & W_{L/\mathbb{Q}} & \to & \mathrm{Gal}(L/\mathbb{Q}) \quad (\mathbb{Q}) \end{array}$$

And for $\nu = \infty$, p and $\bar{\nu}_0 = \nu^{\sim}$, $\wp$ we choose a set S^{ν} of representatives in the cosets $\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{\bar{\nu}0}/\mathbb{Q}_{\nu})$ (such that $1 \in S^{\nu}$) and a section $\{\omega^{\nu}_{\tau} \mid \tau \in S^{\nu}\}$ of $W_{L/\mathbb{Q}} \to \mathrm{Gal}(L/\mathbb{Q})$ on S^{ν} (such that $\omega^{\nu}_1 = 1$), and we define a splitting $\delta \to \omega^{\nu}_{\delta}$ of ($\mathbb{Q}$) by $\omega^{\nu}_{\delta} = \omega^{\nu}_{\tau}\, d^{\nu}_{\delta}$ if $\delta = \tau\sigma$ ($\tau \in S^{\nu}$, $\sigma \in \mathrm{Gal}(L_{\bar{\nu}0}/\mathbb{Q}_{\nu})$). If $\{A^{\nu}_{\delta,\sigma}\}$ is the 2-cocycle defined by this splitting, $\{A^{\infty}_{\delta,\sigma}\}$ and $\{A^{p}_{\delta,\sigma}\}$ are cohomologues, then $A^{\infty}_{\delta,\sigma}(A^{p}_{\delta,\sigma})^{-1} = B_{\delta}\delta(B_{\sigma})B_{\delta\sigma}^{-1}$ for a 1-cochain $\{B_{\sigma}\}$ in C_L. $\chi_{\nu} \in X_*(Q(L, m))$ is left fixed by $\mathrm{Gal}(L_{\bar{\nu}0}/\mathbb{Q}_{\nu})$, and we have

$$\Sigma\, \sigma\chi_{\infty} \text{ (sum over } \sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{\nu\sim}/\mathbb{R}))$$
$$= - \Sigma\, \sigma\chi_{p} \text{ (sum over } \sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L_{\wp}/\mathbb{R})).$$

If we let η denote this cocharacter of $Q(L, m)$, the 1-cochain $\{E_{\sigma}\}$ in $C_L \otimes X_*(Q(L, m))$ defined by

$$E_{\sigma} = (\Pi\, (A^{\infty}_{\sigma,\tau})^{\sigma\tau\chi\infty})(\Pi\, (A^{p}_{\sigma,\tau})^{\sigma\tau\chi p})B^{\eta}_{\sigma}$$
$$\text{(product over } \tau \in S^{\infty}, S^{p})$$

satisfies

$$E_{\delta}\delta(E_{\sigma})E_{\delta\sigma}^{-1} = D^{\infty}_{\delta,\sigma}D^{p}_{\delta,\sigma},$$

where $D^{\nu}_{\delta,\sigma} \in \Pi_{\bar{\nu}|\nu}\, Q(L, m)(L_{\bar{\nu}})$ is defined by $D^{\nu}_{\delta,\sigma} = \Pi_{\bar{\nu}|\nu}\, \tau''((d^{\nu}_{\delta\tau'',\sigma\tau'})^{\chi\nu})$, here $\tau, \tau', \tau'' \in S^{\nu}$ and $\delta_{\tau''}, \delta_{\tau'} \in \mathrm{Gal}(L_{\bar{\nu}0}/\mathbb{Q}_{\nu})$ are given by: τ is the element in S^{ν} associated to $\bar{\nu}$ (that is $|\tau x|_{\bar{\nu}} = |x|_{\bar{\nu}0}$), $\sigma\tau = \tau'\sigma_{\tau'}$ and $\delta\tau' = \tau''\delta_{\tau''}$, τ'' denotes also the isomorphism $Q(L, m)(L_{\bar{\nu}0}) \leftrightarrow Q(L,m)(L_{\bar{\nu}''})$ defined by τ''.

Now if $e_\sigma \in Q(L, m)(\mathbb{A}_L)$ is a lifting of E_σ (with respect to the projection $Q(L, m)(\mathbb{A}_L) \to C_L \otimes X_*(Q(L, m)))$, we have

$$e_\delta \delta(e_\sigma) e_{\delta\sigma}^{-1}\, t_{\delta,\sigma} = D^\infty{}_{\delta,\sigma} D^p{}_{\delta,\sigma},$$

for a 2-cocycle $\{t_{\delta,\sigma}\}$ in $Q(L, m)(L)$, this 2-cocycle defines an exact sequence

$$Q(L, m)(L) \to \mathcal{L}^L{}_{L,m} \to \mathrm{Ga1}(L/\mathbb{Q})$$

with a splitting $\sigma \to t_\sigma \in \mathcal{L}^L{}_{L,m}$ (that is $t_\delta t_\sigma t_{\delta\sigma}^{-1} = t_{\delta,\sigma}$), and $\{e_v\}$ defines a homomorphism ζ_v of exact sequences

$$\begin{array}{ccccc} L^\times{}_{\bar v 0} & \to & W_{L\bar v 0/\mathbb{Q}v} & \to & \mathrm{Gal}(L_{\bar v 0}/\mathbb{Q}_v) \\ \downarrow & & \downarrow & & \downarrow \\ Q(L, m)(L_{\bar v 0}) & \to & (\mathcal{L}^L{}_{L,m})_{\bar v 0} & \to & \mathrm{Gal}(L_{\bar v 0}/\mathbb{Q}_v) \end{array}$$

by χ_v on the kernel and $d_\sigma \to (e_\sigma | Q(L, m)(L_{\bar v 0}))t_\sigma$, and, for $\ell \neq p$ and imbedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$, a splitting ζ_ℓ of

$$Q(L, m)(L_{\bar{\wp}}) \to (\mathcal{L}^L{}_{L,m})_{\bar{\wp}} \to \mathrm{Gal}(L_{\bar{\wp}}/\mathbb{Q}_\ell)$$

by $\sigma \to (e_\sigma | Q(L, m)(L_{\bar{\wp}}))t_\sigma$, here $\bar{\wp}$ is the prime ideal of L defined by $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$.

$\mathcal{L}^L{}_{L,m}$ is uniquely determined up to an isomorphism which transforms these local homomorphisms into equivalent.

By forward and backward transform we have a gerb

$$Q(L, m)(\overline{\mathbb{Q}}) \to \mathcal{L}^L{}_m \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

and local homomorphisms ζ_v: $\mathcal{D}^L{}_{\bar v 0} \to \mathcal{L}^L{}_m$ ($v = \infty$, p), ζ_ℓ: $\mathcal{G}_\ell \to \mathcal{L}^L{}_m$ ($\ell \neq p$).

For $L \subset L'$ ($\subset \overline{\mathbb{Q}}$) and m|m' we have a homomorphism $\mathcal{L}^{L'}{}_{m'} \to \mathcal{L}^L{}_m$ transforming χ'_v to $[L'_{\bar v'' 0}:L_{\bar v 0}]\chi_v$ ($v = \infty$, p), therefore we have a limit $\mathcal{L} \leftarrow {}_{L,m}\lim \mathcal{L}^L{}_m$ and local homomorphisms ζ_∞: $\mathcal{W} \to \mathcal{L}$, ζ_p: $\mathcal{D} \to \mathcal{L}$ and ζ_ℓ: $\mathcal{G}_\ell \to \mathcal{L}$.

Definition of $\xi^{\infty}{}_{\mu}: \mathcal{W} \to G_T$ and $\xi^{p}{}_{\mu}: \mathcal{D} \to G_T$

Let ν be a place (of $\mathbb{Q}$) and let $\overline{\mathbb{Q}}_\nu$ be an algebraic closure of $\mathbb{Q}_\nu$. Let T be a $\mathbb{Q}_\nu$-torus which splits over the Galois extension L ($\subset \overline{\mathbb{Q}}_\nu$) of $\mathbb{Q}_\nu$ and let $\mu \in X_*(T)$.

We define a homomorphism ξ_μ of exact sequences

$$\begin{array}{ccccc} L^\times & \to & W_{L/\mathbb{Q}\nu} & \to & \mathrm{Gal}(L/\mathbb{Q}_\nu) \\ \downarrow & & \downarrow\xi_\mu & & \downarrow \\ T(L) & \to & T(L)\times\mathrm{Gal}(L/\mathbb{Q}_\nu) & \to & \mathrm{Gal}(L/\mathbb{Q}_\nu) \end{array}$$

by $\Sigma\ \sigma\mu$ (sum over $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_\nu)$) on the kernel and $d_\sigma \to \Pi\ (d^\nu{}_{\sigma,\delta})^{\sigma\delta\mu}\times\sigma$ (product over $\delta \in \mathrm{Gal}(L/\mathbb{Q}_\nu)$).

By forward and backward transform we have a homomorphism of gerbs $\xi_\mu: \mathcal{D}^L \to G_T$ and by going to limit we have a homomorphism of gerbs $\xi_\mu: \mathcal{D}^\nu \to G_T$.

Definition of ψ_μ: $\mathscr{L} \rightarrow \mathcal{G}_T$

Let T be a $\mathbb{Q}$-torus which splits over the Galois extension L ($\subset \overline{\mathbb{Q}}$) of $\mathbb{Q}$ and let $\mu \in X_*(T)$. For $m \in \mathbb{N}$ sufficiently large we define a homomorphism ψ_μ: $Q(L, m) \rightarrow T$ defined over $\mathbb{Q}$ in the following way: choose $a \in L^\times$ such that $(a) = \wp^r$ (some $r \in \mathbb{N}$) ($\wp$ is the prime ideal of L defined by $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_\ell$) and $|Nm_{L_\wp/\mathbb{Q}_p} a| = q$ ($= p^m$), then

$$\gamma = \Pi_{\sigma\in Gal(L/\mathbb{Q})}\sigma(a)^{\sigma\mu} \ (\in T(L))$$

belongs to $T(\mathbb{Q})$ and for $\lambda \in X^*(T)$, $\lambda(\gamma)$ belongs to Y(L, m), therefore γ defines a homomorphism $X^*(T) \rightarrow X^*(Q(L, m))$ which commutes with the action of Gal $(L/\mathbb{Q})$, then ψ_μ is the homomorphism defined by this homomorphism of character groups.

For k ($\in \mathbb{N}$) sufficiently large we can find a section s of the projection χ: $Y(L, m) \rightarrow X^*(Q(L, m))$ on $kX^*(Q(L, m))$ commuting with the action of $Gal(L/\mathbb{Q})$, and for n ($\in m k\mathbb{N}$) sufficiently large we can find a $\delta_n \in Q(L, m)(\mathbb{Q})$ satisfying $\chi_\pi(\delta_n) = s(k\chi_\pi)^{n/mk}$ for every $\pi \in Y(L, m)$. δ_n is not uniquely determined, but $\chi_\pi(\delta_n)\pi^{-n/m}$ is an unit for each π. $\{\delta_n^j \mid j \in \mathbb{Z}\}$ is Zariski dense in $Q(L, m)(\mathbb{Q})$.

ψ_μ is characterized by $\psi_\mu(\delta_{mn}) = \gamma^n$ modulo an unit.

Now we will extend ψ_μ to a homomorphism of gerbs ψ_μ: $\mathscr{L} \rightarrow \mathcal{G}_T$.

If, for $\nu = \infty$, p,

$$E^\nu_\sigma = \Pi\ \Pi\ (A^\nu_{\sigma\tau,\delta})^{\sigma\tau\delta\mu} \in C_L \otimes X_*(T)$$
$$\text{(products over } \tau \in S^\nu,\ \delta \in Gal(L_{\bar{\nu}0}/\mathbb{Q}_\nu))$$

and

$$F = \Pi\ B^{-\delta\mu}_\delta \in C_L \otimes X_*(T) \text{ (product over } \delta \in Gal(L/\mathbb{Q})),$$

then E^{v}_{σ} belongs to $\Pi_{\bar{v}|v}\, T(L_{\bar{v}})$ and we have

$$\psi_{\mu}(E_{\sigma}) = e_{\sigma}'\, F\sigma(F)^{-1}$$

where $e_{\sigma}' = E^{\infty}{}_{\sigma} E^{p}{}_{\sigma}{}^{-1}$, and if $f \in T(\mathbb{A}_{L})$ is a lifting of F, then

$$\psi_{\mu}(e_{\sigma}) = s_{\sigma}{}^{-1} e_{\sigma}' f\sigma(f)^{-1},$$

where $s_{\sigma} \in T(L)$. The 1-cochain $\{s_{\sigma}\}$ satisfies $s_{\delta}\delta(s_{\sigma})s_{\delta\sigma}{}^{-1} = \psi_{\mu}(t_{\rho,\sigma})$ and we define the remaining part of ψ_{μ} (on $\mathscr{L}^{L}{}_{L,m}$) by $t_{\sigma} \rightarrow s_{\sigma}\times\sigma$. By going to limit we have a homomorphism ψ_{μ}: $\mathscr{L} \rightarrow G_{T}$ of gerbs, it is determined up to composition with an automorphism of G_{T} which is locally equivalent to the identical automorphism.

We have equivalences

$$\psi_{\mu}\circ\zeta_{\infty} \sim \xi^{\infty}{}_{\mu},\ \psi_{\mu}\circ\zeta_{p} \sim \xi^{p}{}_{-\mu} \text{ and } \psi_{\mu}\circ\zeta_{\ell} \sim \xi_{\ell}\ (\ell \neq p),$$

because

$e'_{\sigma}|T(L_{v\sim}) = \Pi\, (A^{\infty}{}_{\sigma,\delta})^{\sigma\delta\mu}$ (product over $\delta \in Gal(L_{v\sim}/\mathbb{R})$)

$e'_{\sigma}|T(L_{\wp}) = \Pi\, (A^{p}{}_{\sigma,\delta})^{-\sigma\delta\mu}$ (product over $\delta \in Gal(L_{\wp}/\mathbb{Q}_{p})$)

$e'_{\sigma}|T(L_{\bar{\wp}}) = 1$ for $\bar{\wp}|\ell$, $\ell \neq p$.

Definition of $\wp$ and $\mathscr{L} \to \wp$

If we in the definition of Y(L, m) figuring in the definition of $\mathcal{L}$ replace the quantity

$$|\Pi\ \sigma\pi|^{[L:\mathbb{Q}]^{\wedge}(-1)} \text{ (product over } \sigma \in \mathrm{Gal}(L/\mathbb{Q}))$$

by

$$|\Pi\ \sigma\pi|^{[L_\nu:\mathbb{Q}]^{\wedge}(-1)} \text{ (product over } \sigma \in \mathrm{Gal}(L_\nu/\mathbb{Q})),$$

and in the definition of Y*(L, m) replace

{unity in Y(L, m)} by {roots of unity in Y(L, m)},

then we get a new exact sequence and a homomorphism:

$$\begin{array}{ccccc} Q(L, m)(L) & \to & \mathscr{L}^L{}_{L,m} & \to & \mathrm{Gal}(L/\mathbb{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ P(L, m)(L) & \to & \wp^L{}_{L,m} & \to & \mathrm{Gal}(L/\mathbb{Q}), \end{array}$$

and by forward and backward transform and then going to limit, we get a gerb $\wp$ and a homomorphism $\mathscr{L} \to \wp$.

A homomorphism ψ_μ: $\mathcal{L} \to G_T$ as above factorizes through $\mathscr{L} \to \wp$ if $\mu \in X_*(T)$ satisfies the Serre condition:

$$(\sigma - 1)(\iota + 1)\mu = (\iota + 1)(\sigma - 1)\mu = 0$$

for each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ (ι is the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$).

The elements $\delta_n \in P(L, m)(\mathbb{Q})$ (n sufficiently large multiple of m) are now uniquely determined and $\chi_\pi(\delta_n) = \pi^{n/m}$ for $\pi \in Y(L, m)$, also $\psi_\mu|P(L, m)(\mathbb{Q})$ is characterized by $\psi_\mu(\delta_n) = \gamma^{n/m}$.

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